Quantile Impulse Response Analysis with Applications in Macroeconomics and Finance

Whayoung Jung*   Ji Hyung Lee†

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Abstract

This paper studies quantile impulse response functions (QIRFs) and their applications in macroeconomics and finance. We build a multi-equation autoregressive conditional quantile model and propose a new construction and statistical inference of the QIRF. We investigate dynamic QIRFs of the US economy in response to monetary policy and financial shocks, providing some interesting results: (i) Economic activity has the most heterogeneous response across its distribution among the variables under study. The left tail of economic activity is the most responsive to monetary policy and financial stimuli. (ii) The conditional 5% quantile of economic activity (*Growth-at-Risk*) shows a much more persistent response to a monetary policy shock than the mean IRF of the economic activity. A financial shock, on the contrary, has an acute but transient impact on Growth-at-Risk. (iii) We also assess the impacts of financial and monetary policy shocks on Growth-at-Risk during the global financial crisis. Negative financial shocks during August 2007–June 2009 substantially aggravated Growth-at-Risk over 2008–2009. Unconventional monetary policy tools used during July 2009–December 2015 ameliorated Growth-at-Risk successfully over 2010–2015. (iv) When a measure of financial conditions (NFCI) stays at its right tail quantiles (tighter financial conditions), NFCI displays a locally explosive behavior. As a result, the consecutive right tail events create extreme downside risks to the economy. The tool set of QIRFs, therefore, provides detailed dynamic distributional evolution of macroeconomic and financial variables over time in response to economic shocks.

Keywords: Financial Crisis, Growth-at-Risk, Monetary Policy, Quantile Impulse Response

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*Department of Economics, University of Illinois, wjung16@illinois.edu.
†Department of Economics, University of Illinois, jihyung@illinois.edu.

1 Introduction

The conditional mean of an outcome variable has been the primary object of study in economics, as it summarizes the central response to explanatory variables. The scientific interests of policymakers and researchers, however, go beyond the conditional mean. Extreme events and business cycles have significant effects on the economy, so we also need to study the tail or shoulder of the outcome distribution. It is therefore important to obtain a complete picture of the dynamic responses of conditional distribution of economic variables.

As an alternative to the conventional mean-regression analysis, Koenker and Bassett (1978) proposed quantile regression (QR). Since QR estimates heterogeneous regression coefficients for a response variable across its conditional distribution, it provides a richer interpretation in regression analysis (see Koenker (2005) for the textbook treatment). Recent development in time series QR models enables researchers to study the dynamics at various parts of an outcome distribution.

This paper investigates how quantiles of endogenous variables respond over time to a shock in macro/finance models. We build a QR model accommodating important dynamics of macro/financial time series data. Certain cross-sectional and time series characteristics, such as *signed* dispersion and persistence, are important in their distributional evolution over time. As those characteristics are not fully captured by observable variables, we modify an autoregressive conditional quantile specification. In particular, we model the conditional quantile of response variable as a function of past observations as well as their own past quantiles. We adopt the CAViaR-type model by Engle and Manganelli (2004), but extend it to include the level impact of macro variables as well as the asymmetric signed dispersion effect (see Remark 2.1 below for a detailed discussion).

Using the proposed multi-equation QR model, we construct quantile impulse response functions (QIRFs). Since the Wold moving average representation is not available in the QR model, the QIRF is not defined in the same way as the mean impulse response function (IRF). Therefore, we suggest an alternative definition and construction of the QIRF. This QIRF provides a heterogeneous shock-response mechanism across the distribution, complementing the conventional mean IRF analysis in macro/finance.

For empirical applications, we investigate QIRFs of US macroeconomic variables to monetary policy and financial shocks. Our main findings follow. Firstly, we find that economic activity has the most heterogeneous response across quantiles, while the response of financial variables is relatively homogeneous. An expansionary monetary policy shock shifts the distribution of economic activity to the right. The shock significantly reduces downside risk to growth but merely affects upside risk. On the contrary, a financial shock shifts the economic activity distribution to the left. The left tail quantiles are substantially more responsive than the median or upper quantiles. This empirical result is in line with Adrian, Boyarchenko, and Giannone (2019), who show that deteriorating financial conditions strongly increase the downside risks to growth but not the upside risks. Moreover, a monetary policy shock has much more persistent effects on the conditional 5% quantile of economic activity (*Growth-at-Risk*) than on its mean. On the contrary, a financial shock has an acute but transient impact on Growth-at-Risk.

Secondly, we quantitatively assess how much downside and upside risks to growth were affected by the financial and monetary policy shocks during the global financial crisis. Financial shocks during August 2007–June 2009 substantially decreased the 5% quantile of Chicago Fed National Activity Index (CFNAI) by 1.7 on average over 2008–2009.[1] However, the decrease in its 95% quantile due to the shocks over the same period was much less: -0.5. Monetary policy shocks during July 2009–December 2015 increased the 5% quantile by 0.5 on average over 2010–2015. The increase suggests the unconventional monetary measures effectively reduced downside risks. On the other hand, the upper quantile was increased merely by 0.1.

Thirdly, a measure of financial conditions (NFCI) exhibits locally explosive dynamics when it stays at its right tail quantiles (tighter financial conditions).[2] Accordingly, a series of its right tail events lead to a sharp deterioration of financial conditions, which creates substantial downside risk to the economy. This locally explosive behavior of financial conditions highlights how a financial crisis can develop in a short period of time.

This paper relates to several strands of literature. Over the past few decades, various times series QR models have been developed to describe heterogeneous dynamics at different parts of the conditional distribution. Koenker and Zhao (1996), Xiao and Koenker (2009), Koenker and Xiao (2006), and Xiao (2009) estimate QR models for autoregressive conditional heteroskedasticity (ARCH), generalized ARCH (GARCH), autoregressive (AR), and cointegrated processes, respectively. Using QR methods, they relax the traditional constant-coefficient dynamics and study asymmetric dynamics in economic time series.

While the conditional quantile is modeled as a linear function of past observations in the above models, Engle and Manganelli (2004) develop autoregressive conditional quantile specifications. In their univariate QR model, the conditional quantile depends not only on past observations but also on unobservable past conditional quantiles. White, Kim, and Manganelli (2010) extend the univariate model to a multi-quantile model, and White, Kim, and Manganelli (2015, WKM henceforth) further extend the model to multivariate and multi-quantile models. We adopt their autoregressive conditional quantile specification in our QR model. With that specification, latent information such as dispersion and persistence of distribution is effectively incorporated to the evolution of the conditional distribution. While there are QR models studying tail dependence across variables (see, e.g., WKM; Li, Li, and Tsai 2015; Adrian and Brunnermeier 2016, Han, Linton, Oka, and Whang 2016), we do not allow for such co-dependence.[3] See also Xiao (2012) and Linton and Xiao (2017) for recent advances in time series QR models and applications.

In terms of empirical applications, our paper is related to recent literature studying asymmetric effects of economic state variables on Growth-at-Risk. Adrian, Boyarchenko, and Giannone (2019) find that the lower quantiles of economic activity are substantially affected by financial conditions, while the upper quantiles are not. Using a different data set and a different QR model, our

[1] CFNAI is a monthly index for US economic activity. Section 5.1 explains the index in detail.

[2] NFCI is an index for US financial conditions. A higher NFCI represents tighter financial conditions. See Section 5.1 for more details about the index.

[3] See Remark 2.2 for a detailed explanation.

empirical findings are consistent with theirs. With the QIRF, we further describe the evolution of the conditional quantile over time in response to a shock. Adrian, Grinberg, Liang, and Malik (2018) study how the effects of financial conditions on Growth-at-Risk evolve over time. They show that looser financial conditions increase the lower quantiles of GDP growth in the short run, but decrease the lower quantiles in the medium term. While they estimate a single equation explaining the conditional quantile using the local projection method, our approach differs as the QIRF is constructed based on a multi-equation describing the evolution of the system. In our empirical analysis, we do not find a statistically significant reversal in the QIRF of economic activity to a financial shock. Loria, Matthes, and Zhang (2019) study how Growth-at-Risk responds to various shocks with local projections as well. They show that the lower quantiles of GDP growth are affected by all shocks (monetary policy, credit spread, and productivity shocks) under study more than other quantiles. Their empirical results are also in line with our application results.

The QR methods have been also used to study heterogeneous dynamics for macroeconomic and financial variables. Chevapatrakul, Kim, and Mizen (2009) evaluate the quantile response of interest rates to inflation and the output gap. Galvao, Montes-Rojas, and Park (2013) find that house price returns in the UK have heterogeneous persistence and asymmetric responses to income and interest rates across quantiles. Mumtaz and Surico (2015) show that the dynamic relationship between consumption and interest rates changes depending on the conditional quantile of consumption.

More recently, there have been a few papers constructing QIRFs to describe the evolution of quantile responses over time against a shock. The pseudo-QIRF of WKM estimates the response of Value-at-Risk (VaR) to a shock. Their QIRF underestimates the magnitude of a shock as the volatility processes are ignored in the QIRF construction. The QIRF of Montes-Rojas (2019) describes the cumulative impact of a series of shocks, not a one-off shock, because persistent realizations of lower (or upper) quantiles are assumed in its construction. As a result, his QIRF is not comparable to the IRF which measures the mean response to a one-time shock. The QIRF of Han, Jung, and Lee (2019) measures the expected change in the conditional quantile due to a shock. However, estimation of their QIRF is not effective for data with nontrivial first moment dynamics, such as macroeconomic variables, since they use the QIRF local projection estimator. The QIRF of Kim, Lee, and Mizen (2019) is conceptually similar with that of Han, Jung, and Lee (2019), measuring the expected change in the conditional quantile after a shock. But, their QIRF is estimated by combining the mean-based IRF and parameters in the quantile model.[4]

Compared to the above QIRFs, the QIRF suggested in this paper properly measures the impact of a one-off shock, thus is comparable to the standard IRF. We do not assume specific first or second moment dynamics for variables of interest. Moreover, we estimate QIRFs using the QR model estimation once the structural shock is identified. Section 3.3 carefully compares our QIRF to the existing QIRFs in recent studies.

[4]Chavleishvili and Manganelli (2019) employ *the law of iterated quantiles* for defining their QIRF. They (p.10) explain that the QIRF describes the impact of a shock on the quantiles of the distribution of future quantiles.

The rest of the paper is organized as follows. Section 2 introduces our QR model, and Section 3 proposes the definition and construction of the QIRF, with a brief comparison to the recent literature. Section 4 discusses estimation of the model and QIRF, then provides inferential methods based on asymptotic theory. In Section 5, we study QIRFs of US economy. In particular, we study the dynamic quantile responses of economic activity during the global financial crisis. We also assume a distress scenario of a series of financial market tail events, and examine the quantile responses.

2 Model Framework

In this section, we introduce the autoregressive conditional quantile model in a multivariate framework. We also briefly compare our model with the related time series QR models in other recent papers.

2.1 Quantile Regression Model

Let the vector of variables of interest be $\mathbf{y}_t = [y_{1t}\ y_{2t}\ \ldots\ y_{nt}]^\top$ whose dimension is n by 1, and define a natural filtration $\{\mathcal{F}_t\}_{t\geq 1}$. For $i = 1, 2, ..., n$ and $\tau \in (0, 1)$, $Q_{y_{it}}(\tau|\mathcal{F}_{t-1})$ denotes the τ-quantile of y_{it} conditional on $\mathcal{F}_{t-1}$ such that $Pr(y_{it} \leq Q_{y_{it}}(\tau|\mathcal{F}_{t-1})\big|\mathcal{F}_{t-1}) = \tau$. For notational simplicity, let $q_{it}(\tau) := Q_{y_{it}}(\tau|\mathcal{F}_{t-1})$.

Our QR model studies the evolution of the conditional distribution using the autoregressive conditional quantile specifications. For variable i, its conditional τ-quantile is modeled as a function of $\mathbf{y}_{t-1}$ and its own past quantiles:

$$q_{it}(\tau) = c_{i\tau} + \mathbf{a}_{i\tau}^\top \mathbf{y}_{t-1} + \sum_{k=1}^{p} [b_{k,i\tau} q_{it-k}(0.5) + d_{k,i\tau} q_{it-k}(\tau)], \tag{1}$$

where $d_{k,i0.5} = 0$. The vector coefficient $\mathbf{a}_{i\tau}$ measures how the conditional quantile responds to the collection of the previous economic variables $\mathbf{y}_{t-1}$. The summation terms in (1) describe autoregressive dynamics along its own quantiles.

Remark 2.1 *Observe that*

$$b_{k,i\tau} q_{it-k}(0.5) + d_{k,i\tau} q_{it-k}(\tau) = b_{k,i\tau}[q_{it-k}(0.5) - q_{it-k}(\tau)] + (b_{k,i\tau} + d_{k,i\tau}) q_{it-k}(\tau),$$

so that $b_{k,i\tau}$ and $b_{k,i\tau} + d_{k,i\tau}$ represent the effect of dispersion of the conditional distribution and the quantile persistence, respectively. As a measure of dispersion, we use the distance between the conditional τ-quantile and the median. To get the idea, consider a scale model of $y_t = \sigma_t \varepsilon_t$ with $\sigma_t \in \mathcal{F}_{t-1}$ and $\varepsilon_t \sim iid\ F_\varepsilon$, then it is easy to show that

$$q_{it-k}(0.5) - q_{it-k}(\tau) = \left(F_\varepsilon^{-1}(0.5) - F_\varepsilon^{-1}(\tau)\right) \sigma_t,$$

where $F_\varepsilon^{-1}(\cdot)$ is the inverse CDF. Therefore, $q_{it-k}(0.5)-q_{it-k}(\tau)$ is a volatility scaled by $F_\varepsilon^{-1}(0.5)-F_\varepsilon^{-1}(\tau)$ whose sign is positive for $\tau < 0.5$, and negative for $\tau > 0.5$. Naturally, we label the term $q_{it-k}(0.5) - q_{it-k}(\tau)$ as the signed dispersion. Each quantile depends on a different conditional dispersion measure, with an opposite sign between left- and right-tails, and persistence. Thus, the proposed QR model provides rich flexibility in modeling the evolution of the conditional distribution. The lagged conditional quantiles incorporate information not captured by observable variables. Persistence and dispersion play important roles in distribution dynamics in practice. Thus, we use the past conditional quantiles to include the signed dispersion and autoregressive terms (persistence). Moreover, observed economic variables introduce the important mean level effect of macro time series. These specifications also allow for a smooth change of the conditional quantile and help reduce the number of parameters.

Let $\mathbf{q}_t(\tau) = [q_{1t}(\tau)\ q_{2t}(\tau)\ \ldots\ q_{nt}(\tau)]^\top$. Define a n by 1 matrix $\mathbf{c}(\tau) = [c_{1\tau}\ c_{2\tau}\ \ldots\ c_{n\tau}]^\top$ and n by n matrices

$$\mathbf{A}(\tau) = \begin{bmatrix} \mathbf{a}_{1\tau}^\top \\ \mathbf{a}_{2\tau}^\top \\ \vdots \\ \mathbf{a}_{n\tau}^\top \end{bmatrix},\ \mathbf{B}_k(\tau) = \begin{bmatrix} b_{k,1\tau} & 0 & \ldots & 0 \\ 0 & b_{k,2\tau} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b_{k,n\tau} \end{bmatrix},\ \mathbf{D}_k(\tau) = \begin{bmatrix} d_{k,1\tau} & 0 & \ldots & 0 \\ 0 & d_{k,2\tau} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_{k,n\tau} \end{bmatrix}.$$

With the matrix notation, a multi-equation system of (1) can be concisely expressed as

$$\mathbf{q}_t(\tau) = \mathbf{c}(\tau) + \mathbf{A}(\tau)\mathbf{y}_{t-1} + \sum_{k=1}^{p}[\mathbf{B}_k(\tau)\mathbf{q}_{t-k}(0.5) + \mathbf{D}_k(\tau)\mathbf{q}_{t-k}(\tau)], \tag{2}$$

where $\mathbf{D}_k(0.5) = \mathbf{0}$.

Remark 2.2 *Some QR models allow the quantile dependence across variables. For example, WKM study tail dependence between variables for daily stock returns data (spillovers in VaR). However, our model does not allow interaction between quantiles of different variables. The first main reason is that the frequency of macroeconomic data is relatively low, such as quarterly or monthly observations. If we allow non-zero off-diagonal entries in $\mathbf{B}_k(\tau)$ and $\mathbf{D}_k(\tau)$, the estimation becomes infeasible due to the small number of observations relative to the number of parameters. Secondly, the macroeconomic variables show long-term fluctuation and co-dependence, so the degree of tail dependence across variables is lower than high frequency financial time series. Thus, $\mathbf{B}_k(\tau)$ and $\mathbf{D}_k(\tau)$ in (2) are assumed to be diagonal matrices in our model.*

2.2 Comparison to Related Literature

Some recent studies have expanded the scope of QR time series models and their applications. This section compares our QR model to other related models in the literature.

From the perspective of econometrics, our model is closely related to CAViaR by Engle and Manganelli (2004), MQ-CAViaR by White, Kim, and Manganelli (2010), and VAR for VaR model by WKM. The three papers developed autoregressive conditional quantile specifications in which the conditional quantile depends on the lagged conditional quantiles as well as observable variables. The main applications of their QR models are descriptions of VaR of financial assets. With the GARCH model insight, the above three models specify stock returns' quantile dynamics based on their volatility processes. Our model, on the other hand, aims to encompass the conditional quantile modeling of macroeconomic variables, without a specific volatility process assumption. Thus, our QR model is well suited to empirical applications in both macroeconomics and finance.

In terms of empirical applications, our model is closely related to recent QR models for macroeconomic variables. Adrian, Boyarchenko, and Giannone (2019), Adrian, Grinberg, Liang, and Malik (2018), and Loria, Matthes, and Zhang (2019) estimate how Growth-at-Risk responds to observed economic variables. Montes-Rojas (2019), Kim, Lee, and Mizen (2019), and Chavleishvili and Manganelli (2019) estimate multi-equation systems for the conditional quantile of endogenous variables and construct QIRFs, though they differ on structural identification.[5] In the above QR models, the conditional quantile is a linear function of observable variables only. However, our model incorporates unobservable information with autoregressive conditional quantile specifications, which allows for a richer dimension in the evolution of distribution.

3 Quantile Impulse Response Function

In this section, we construct QIRFs to investigate how the conditional quantile of an outcome variable responds to a shock over time. If the shock only affects the location of an outcome distribution, quantile responses will be homogeneous across quantiles. If the shock changes the entire shape of the distribution, however, QIRFs will show heterogeneous impacts on each quantile. The complete picture of the QIRF mechanism, therefore, complements the conventional IRF that measures the impact of a shock only on the conditional mean of economic variables.

In the mean-based VAR model, IRF is defined as $\frac{\partial \mathbf{y}_{t+s}}{\partial \boldsymbol{\epsilon}_t'}$ with a structural innovation $\boldsymbol{\epsilon}_t$, which can be constructed from the Wold moving average (MA) representation. However, the QIRF can not be defined in the same way as QR models do not have the fundamental MA representation. Thus, we propose an alternative definition of the QIRF using the structure of the QR model. We measure the quantile response by comparing the conditional quantiles from two different dynamic paths, one of which is hit by a shock and the other of which is not.

[5]Chavleishvili and Manganelli (2019) estimate a recursive quantile vector autoregressive (VAR) model on which a Cholesky identification is directly imposed. That is, a structural shock is identified in the QR model. However, Montes-Rojas (2019) and Kim, Lee, and Mizen (2019) use the mean-based VAR model to identify a structural shock since their multivariate quantile models are reduced-form.

3.1 Heuristics on QIRF

This section discusses QIRFs in a heuristic way using the imposed model structure. At horizon $s = 1$, the shock $\boldsymbol{\epsilon}_t$ affects $\mathbf{q}_{t+1}(\tau)$ only through its previous realization. From (2), the change in the conditional τ-quantile due to the shock is

$$Q_{\mathbf{y}_{t+1}}(\tau \mid \tilde{\mathbf{y}}_t; \mathcal{F}_{t-1}) - Q_{\mathbf{y}_{t+1}}(\tau \mid \mathbf{y}_t; \mathcal{F}_{t-1}) = \mathbf{A}(\tau)[\tilde{\mathbf{y}}_t - \mathbf{y}_t] = \mathbf{A}(\tau)\boldsymbol{\epsilon}_t,$$

where $\tilde{\mathbf{y}}_t = \mathbf{y}_t + \boldsymbol{\epsilon}_t$.

From the horizon $s \geq 2$, $\boldsymbol{\epsilon}_t$ affects $\mathbf{q}_{t+s}(\tau)$ through both the previous realization ($\mathbf{y}_{t+s-1}$) and past quantiles $\{\mathbf{q}_{t+s-k}(\tau), \mathbf{q}_{t+s-k}(0.5)\}_{k=1}^{p}$. Since our QR model estimates the effect of a shock on a specific quantile, its impact through past quantiles is easily obtained. At horizon $s = 2$, for example, the change in $\mathbf{q}_{t+2}(\tau)$ attributable to the impact through past quantiles is

$$\begin{aligned}&\mathbf{B}_1(\tau)[Q_{\mathbf{y}_{t+1}}(0.5 \mid \tilde{\mathbf{y}}_t; \mathcal{F}_{t-1}) - Q_{\mathbf{y}_{t+1}}(0.5 \mid \mathbf{y}_t; \mathcal{F}_{t-1})] + \mathbf{D}_1(\tau)[Q_{\mathbf{y}_{t+1}}(\tau \mid \tilde{\mathbf{y}}_t; \mathcal{F}_{t-1}) - Q_{\mathbf{y}_{t+1}}(\tau \mid \mathbf{y}_t; \mathcal{F}_{t-1})]\\ &= [\mathbf{B}_1(\tau)\mathbf{A}(0.5) + \mathbf{D}_1(\tau)\mathbf{A}(\tau)]\boldsymbol{\epsilon}_t.\end{aligned}$$

However, it is not easy to derive the impact of a shock through the previous realization under the QR framework. The effect is

$$\mathbf{A}(\tau)\int_{\tau=0}^{1}[Q_{\mathbf{y}_{t+1}}(\tau \mid \tilde{\mathbf{y}}_t; \mathcal{F}_{t-1}) - Q_{\mathbf{y}_{t+1}}(\tau \mid \mathbf{y}_t; \mathcal{F}_{t-1})]d\tau = \mathbf{A}(\tau)\int_{\tau=0}^{1}\mathbf{A}(\tau)d\tau\boldsymbol{\epsilon}_t,$$

which is difficult to calculate without a distributional assumption, since it requires the expected value. The following section introduces a new definition of QIRF to avoid this infeasible construction.

3.2 Definition and Construction of QIRF

Considering the difficulty in deriving the quantile response, we suggest an alternate definition of QIRF. We construct the QIRF by assuming the realizations of $\{\mathbf{y}_{t+s}\}_{s\geq 1}$ after a shock arrives at time t. Let us consider the following two time paths:

$$\begin{array}{llllllll}\{\ \ldots\ , & \mathbf{y}_{t-2}, & \mathbf{y}_{t-1}, & \mathbf{y}_t, & \mathcal{M}_{t+1}(\mathbf{y}_t), & \mathcal{M}_{t+2}(\mathbf{y}_t), & \mathcal{M}_{t+3}(\mathbf{y}_t), & \ldots\ \},\\ \{\ \ldots\ , & \mathbf{y}_{t-2}, & \mathbf{y}_{t-1}, & \tilde{\mathbf{y}}_t, & \mathcal{M}_{t+1}(\tilde{\mathbf{y}}_t), & \mathcal{M}_{t+2}(\tilde{\mathbf{y}}_t), & \mathcal{M}_{t+3}(\tilde{\mathbf{y}}_t), & \ldots\ \},\end{array} \tag{3}$$

where $\tilde{\mathbf{y}}_t = \mathbf{y}_t + \boldsymbol{\epsilon}_t$ and

$$\mathcal{M}_{t+s}(\mathbf{y}_t) = \begin{cases} Q_{\mathbf{y}_{t+1}}(0.5 \mid \mathbf{y}_t; \mathcal{F}_{t-1}), & \text{for } s = 1,\\ Q_{\mathbf{y}_{t+s}}(0.5 \mid \mathcal{M}_{t+s-1}(\mathbf{y}_t),\ \ldots\ , \mathcal{M}_{t+1}(\mathbf{y}_t), \mathbf{y}_t; \mathcal{F}_{t-1}), & \text{for } s \geq 2.\end{cases}$$

The two paths are identical up to $t-1$. At time t, one path is hit by a shock $\boldsymbol{\epsilon}_t$, but the other is not. After the time t, realizations of $\{\mathbf{y}_{t+s}\}_{s\geq 1}$ are assumed to be the conditional median based on their history for both time paths. When the conditional distribution of y_{it} is symmetric, each of the realization corresponds to the conditional mean. We define the QIRF as the difference between the conditional quantiles from the two time paths.

Definition 3.1

$$QIRF_\tau^{(s)} := \begin{cases} Q_{\mathbf{y}_{t+1}}(\tau \mid \tilde{\mathbf{y}}_t; \mathcal{F}_{t-1}) - Q_{\mathbf{y}_{t+1}}(\tau \mid \mathbf{y}_t; \mathcal{F}_{t-1}), & \textit{for} \;\; s = 1, \\ Q_{\mathbf{y}_{t+s}}(\tau \mid \mathcal{M}_{t+s-1}(\tilde{\mathbf{y}}_t), \; \ldots \; , \mathcal{M}_{t+1}(\tilde{\mathbf{y}}_t), \tilde{\mathbf{y}}_t; \mathcal{F}_{t-1}) & \\ \qquad -Q_{\mathbf{y}_{t+s}}(\tau \mid \mathcal{M}_{t+s-1}(\mathbf{y}_t), \; \ldots \; , \mathcal{M}_{t+1}(\mathbf{y}_t), \mathbf{y}_t; \mathcal{F}_{t-1}), & \textit{for} \;\; s \geq 2. \end{cases}$$

Definition 3.1 can be interpreted as a quantile version of IRF. We follow the intuition by Koop, Pesaran, and Potter (1996) who illustrate that the IRF can be defined as the difference between the conditional means from two different time paths. Those time paths are formulated in the same way as (3) except that realizations of $\{\mathbf{y}_{t+s}\}_{s\geq 1}$ are assumed to be the conditional mean based on their history. We naturally use the conditional median of $\{\mathbf{y}_{t+s}\}_{s\geq 1}$ instead.

Under this definition, QIRFs are recursively expressed as

$$QIRF_\tau^{(s)} = \begin{cases} \mathbf{A}(\tau)\boldsymbol{\epsilon}_t, & \text{for} \;\; s = 1, \\ \mathbf{A}(\tau)QIRF_{0.5}^{(s-1)} + \sum_{k=1}^{r}[\mathbf{B}_k(\tau)QIRF_{0.5}^{(s-k)} + \mathbf{D}_k(\tau)QIRF_\tau^{(s-k)}], & \text{for} \;\; s \geq 2, \end{cases} \tag{4}$$

where $r = \min\{s-1, p\}$. The derivation of $QIRF_\tau^{(s)}$ is in Appendix A.1. The QIRF takes into account the evolution of distribution. For example, a heteroskedastic variable with persistent volatility has a high value of $d_{k,i\tau}$ as explained in Section 2.1. Then, the QIRF accounts for its conditional heteroskedasticity through $\mathbf{D}_k(\tau)$ whose i-th diagonal element is $d_{k,i\tau}$.

As we assume the conditional median for the realizations of $\{\mathbf{y}_{t+s}\}_{s\geq 1}$, the QIRF depends not only on the τ-coefficients, $\{\mathbf{A}(\tau), \mathbf{B}_k(\tau), \mathbf{D}_k(\tau)\}$, but also on the median-coefficients, $\{\mathbf{A}(0.5)$, $\mathbf{B}_k(0.5)$, $\mathbf{D}_k(0.5)\}$ which determine $QIRF_{0.5}^{(s-1)}$.

3.3 QIRF Comparison with Recent Studies

Recently, there have been a few attempts to construct QIRFs based on QR models. However, there is no consensus about the definition of QIRFs. In this section, we compare our QIRF with the related literature.

Similar to our paper, WKM and Montes-Rojas (2019) define their QIRFs as the difference between the conditional quantiles from two time paths: one path is affected by a shock, but the other path (as the benchmark) is not. However, their formulation of the time paths is different from ours. For the pseudo-QIRF of WKM, the two time paths are identical except at the time when a shock hits the system. This scenario does not account for the effect of the shock on subsequent conditional distributions. As a result, their pseudo-QIRF underestimates the magnitude of a shock on quantile responses.[6] Montes-Rojas (2019), on the other hand, assumes persistent realizations of

[6] WKM acknowledge that the pseudo-QIRF ignores the dynamic evolution of distribution. Han, Jung, and Lee

the lower (or upper) quantile for the time paths compared in his QIRF construction.[7] Thus, his QIRF describes the cumulative impact of shocks as if the economy is under continuous distress. Moreover, the QIRFs are not directly comparable across the quantiles since each quantile responds to different shocks.

Compared to the two papers, our QIRF accounts for the impact of a one-time shock assuming realizations of the conditional median after the shock. Montes-Rojas (p.4) also acknowledges that evaluating future realizations at the conditional median is a canonical case. As our QR model employs autoregressive conditional quantile specifications, the QIRF properly accounts for the effects of persistence or dispersion in the distributional evolution.

Han, Jung, and Lee (2019) define QIRFs as the expected change in the conditional quantile in response to a shock. They apply the local projection methods by Jordà (2005) to QR models under the GARCH-type framework, and estimate the expected quantile response directly. However, their estimation of the QIRF is effective in the absence of the first moment dynamics, such as daily stock return data. As the mean effects on macroeconomic variables are not negligible, the estimation method does not work for macroeconomic applications.

The QIRF of Kim, Lee, and Mizen (2019) is conceptually analogous to the QIRF of Han, Jung, and Lee (2019): it measures expected change in the conditional quantile due to a shock. They derive the QIRF based on a structural VAR model in addition to the QR model. Accordingly, their QIRF is constructed by combining the mean-based IRF as well as the QR estimator, while our QIRF only uses the QR estimator.

Adrian, Grinberg, Liang, and Malik (2018) and Loria, Matthes, and Zhang (2019) examine how the quantile response of a dependent variable evolves over time without QIRFs. Instead of constructing QIRFs based on a multi-equation system, they estimate a single equation describing the conditional quantile using local projection estimation.

4 Estimation and Statistical Inference

4.1 Estimation

This section explains estimation of our QR model and QIRF. The estimation of the model is not trivial because of the autoregressive conditional quantile specifications for multi quantiles. We use the following two-step estimation procedure.[8]

Define a $(1+n+2p)$ by 1 vector $\boldsymbol{\theta}_{i\tau} := [c_{i\tau} \quad \mathbf{a}_{i\tau}^{\top} \quad b_{1,i\,\tau} \ \dots \ b_{p,i\,\tau} \ d_{1,i\,\tau} \ \dots \ d_{p,i\,\tau}]^{\top}$ for $i = 1, ..., n$ and $\tau \in (0, 1)$. In the first step, we estimate the median coefficient $\boldsymbol{\theta}_{i\,0.5}$ by solving the following

(2019) discuss and evaluate the performance of the pseudo-QIRF.

[7] In his empirical application, he estimates QIRFs with three variables: the output gap, inflation and the federal funds rate. For the construction of QIRF of output gap at $\tau = 0.1$, for example, the following time paths are assumed. After a shock, the output gap has consecutive realizations at its conditional 10% quantile, but inflation, and the federal funds rate have their realizations at the conditional median.

[8] White, Kim, and Manganelli (2010) also use autoregressive conditional quantile specifications for multi quantiles in the MQ-CAViaR model. In their main application, they adopt a three-step strategy to estimate five different conditional quantiles of the S&P 500 returns.

minimization problem:

$$\min_{\boldsymbol{\theta}_{i\,0.5}} \frac{1}{T}\sum_{t=1}^{T} \rho_\tau(y_{it} - q_{it,0.5}(\boldsymbol{\theta}_{i\,0.5})), \tag{5}$$

where $q_{it,0.5}(\boldsymbol{\theta}_{i\,0.5}) = c_{i\,0.5} + \mathbf{a}_{i\,0.5}^\top \mathbf{y}_{t-1} + \sum_{k=1}^{p} b_{k,i\,0.5} q_{it-k,0.5}(\boldsymbol{\theta}_{i\,0.5})$, $d_{k,i\,0.5} = 0$ and $\rho_\tau(u) = u(\tau - 1[u < 0])$.

In the second step, we estimate the τ-coefficient $\boldsymbol{\theta}_{i\tau}$ based on $\widehat{\boldsymbol{\theta}}_{i\,0.5}$, the estimator from the first step. $\widehat{\boldsymbol{\theta}}_{i\tau}$ solves the minimization problem below:

$$\min_{\boldsymbol{\theta}_{i\tau}} \frac{1}{T}\sum_{t=1}^{T} \rho_\tau(y_{it} - q_{it,\tau}(\boldsymbol{\theta}_{i\tau}|\widehat{\boldsymbol{\theta}}_{i\,0.5})),$$

where $q_{it,\tau}(\boldsymbol{\theta}_{i\tau}|\boldsymbol{\theta}_{i\,0.5}) = c_{i\tau} + \mathbf{a}_{i\tau}^\top \mathbf{y}_{t-1} + \sum_{k=1}^{p}[b_{k,i\tau} q_{it-k,0.5}(\boldsymbol{\theta}_{i\,0.5}) + d_{k,i\tau} q_{it-k,\tau}(\boldsymbol{\theta}_{i\tau}|\boldsymbol{\theta}_{i\,0.5})]$.

Remark 4.1 *Instead of the two-step procedure, $\boldsymbol{\theta}_{i\tau}$ and $\boldsymbol{\theta}_{i\,0.5}$ could be estimated simultaneously. The simultaneous estimation solves the following minimization problem:*

$$\min_{\boldsymbol{\theta}_{i\tau},\boldsymbol{\theta}_{i\,0.5}} \frac{1}{T}\sum_{t=1}^{T} [\rho_\tau(y_{it} - q_{it,\tau}(\boldsymbol{\theta}_{i\tau}, \boldsymbol{\theta}_{i\,0.5})) + \rho_{0.5}(y_{it} - q_{it,0.5}(\boldsymbol{\theta}_{i\,0.5}))],$$

where $q_{it,\tau}(\boldsymbol{\theta}_{i\tau}, \boldsymbol{\theta}_{i\,0.5}) = c_{i\tau} + \mathbf{a}_{i\tau}^\top \mathbf{y}_{t-1} + \sum_{k=1}^{p}[b_{k,i\tau} q_{it-k,0.5}(\boldsymbol{\theta}_{i\,0.5}) + d_{k,i\tau} q_{it-k,\tau}(\boldsymbol{\theta}_{i\tau}, \boldsymbol{\theta}_{i\,0.5})]$ and $d_{k,i0.5} = 0$. Under the simultaneous estimation, a different choice of τ leads to different estimates for $\boldsymbol{\theta}_{i\,0.5}$, though the differences are not substantial. Since the median coefficient $\boldsymbol{\theta}_{i\,0.5}$ plays an important role in the construction of the QIRF, we adopt the two-step estimation strategy which yields robust median coefficient estimates (thus the QIRF). We show the two-step QR estimators are consistent in the following section.

Let $QIRF_{i\tau}^{(s)}$ denote the i-th element of $QIRF_\tau^{(s)}$ in Definition 3.1. From (4), the estimator for $QIRF_{i\tau}^{(s)}$ is recursively constructed using the QR estimator $\widehat{\boldsymbol{\theta}}_{i\tau}$ and $\widehat{\boldsymbol{\theta}}_{i\,0.5}$:

$$\widehat{QIRF}_{i\tau}^{(s)} = \begin{cases} \widehat{\mathbf{a}}_{i\tau}^\top \boldsymbol{\epsilon}_t, & \text{for} \quad s = 1, \\ \widehat{\mathbf{a}}_{i\tau}^\top \widehat{QIRF}_{0.5}^{(s-1)} + \sum_{k=1}^{r}[\widehat{b}_{k,i\tau}\widehat{QIRF}_{i\,0.5}^{(s-k)} + \widehat{d}_{k,i\tau}\widehat{QIRF}_{i\tau}^{(s-k)}], & \text{for} \quad s \geq 2, \end{cases} \tag{6}$$

where $r = \min\{s-1, p\}$.

4.2 Statistical Inference

In this section, we derive the limit theory of the QIRF estimator. Define $\boldsymbol{\Theta}_{i\tau} := [\boldsymbol{\theta}_{1\,0.5}^\top \quad \boldsymbol{\theta}_{2\,0.5}^\top \quad \ldots \quad \boldsymbol{\theta}_{n\,0.5}^\top \quad \boldsymbol{\theta}_{i\tau}^\top]^\top$ and its estimator $\widehat{\boldsymbol{\Theta}}_{i\tau} := [\widehat{\boldsymbol{\theta}}_{1\,0.5}^\top \quad \widehat{\boldsymbol{\theta}}_{2\,0.5}^\top \quad \ldots \quad \widehat{\boldsymbol{\theta}}_{n\,0.5}^\top \quad \widehat{\boldsymbol{\theta}}_{i\tau}^\top]^\top$. We first derive the asymptotic distribution of $\widehat{\boldsymbol{\Theta}}_i$, and then establish the limit theory of the QIRF estimator applying the delta method.

We adopt the modeling assumptions in Section 2 and Appendix of WKM. Consistency and asymptotic normality of the QR estimator follow from the asymptotic theories of Engle and Manganelli (2004) and WKM. $\widehat{\mathbf{\Theta}}_{i\tau}$ is the solution of the following minimization problem:

$$\min_{\mathbf{\Theta}_{i\tau}} \frac{1}{T}\sum_{t=1}^{T}[\rho_\tau(y_{it} - q_{it,\tau}(\boldsymbol{\theta}_{i\tau}|\widehat{\boldsymbol{\theta}}_{i\,0.5})) + \sum_{i=1}^{n}\rho_{0.5}(y_{it} - q_{it,0.5}(\boldsymbol{\theta}_{i\,0.5}))],$$

where $\widehat{\boldsymbol{\theta}}_{i\,0.5}$ is the solution of (5). Let $\epsilon_{it}(\tau) = y_{it} - q_{it}(\tau)$, $\psi_\tau(u) = \tau - 1_{[u\leq 0]}$ and $f_{it,\tau}$ be the density function of $\epsilon_{it}(\tau)$ conditional on $\mathcal{F}_{t-1}$. Define $\nabla q_{it}(\tau)$ as the gradient vector of $q_{it}(\tau)$ with respect to $\mathbf{\Theta}_{i\tau}$.[9] According to the asymptotic theories, the asymptotic distribution of $\widehat{\mathbf{\Theta}}_{\boldsymbol{i\tau}}$ follows.

Lemma 4.1

$$\sqrt{T}(\widehat{\mathbf{\Theta}}_{\boldsymbol{i\tau}} - \mathbf{\Theta}_{\boldsymbol{i\tau}}) \xrightarrow{d} N(\mathbf{0}, \mathbf{Q}^{-1}\mathbf{V}\mathbf{Q}^{-1}), \tag{7}$$

where

$$\mathbf{Q} = E[f_{it,\tau}(0)\nabla q_{it}(\tau)\nabla^\top q_{it}(\tau)] + \sum_{i=1}^{n} E[f_{it,0.5}(0)\nabla q_{it}(0.5)\nabla^\top q_{it}(0.5)],$$
$$\mathbf{V} = E[\boldsymbol{\eta}_t\boldsymbol{\eta}_t^\top],$$
$$\boldsymbol{\eta}_t = \nabla q_{it}(\tau)\psi_\tau(\epsilon_{it}(\tau)) + \sum_{i=1}^{n}\nabla q_{it}(0.5)\psi_{0.5}(\epsilon_{it}(0.5)).$$

Since $\widehat{QIRF}_{i\tau}^{(s)}$ in (6) is a function of $\widehat{\mathbf{\Theta}}_{\boldsymbol{i\tau}}$, we establish the asymptotic distribution of QIRF estimator applying the delta method to (7).

Theorem 4.1

$$\sqrt{T}(\widehat{QIRF}_{i\tau}^{(s)} - QIRF_{i\tau}^{(s)}) \xrightarrow{d} N(\mathbf{0}, \mathbf{G}_{i\tau}^{(s)}\mathbf{Q}^{-1}\mathbf{V}\mathbf{Q}^{-1}\mathbf{G}_{i\tau}^{(s)\top}),$$

where $\mathbf{G}_{i\tau}^{(s)} = \frac{\partial QIRF_{i\tau}^{(s)}}{\partial \mathbf{\Theta}_{\boldsymbol{i\tau}}^{\top}}\Big|_{\widehat{\mathbf{\Theta}}_{i\tau}}$.

$\frac{\partial QIRF_{i\tau}^{(s)}}{\partial \mathbf{\Theta}_{\boldsymbol{i\tau}}^{\top}}$ can be derived recursively, and its derivation is in Appendix A.2. To obtain the variance of QIRF estimator, we use consistent estimators for $\mathbf{Q}$ and $\mathbf{V}$.

Remark 4.2 *Let* $\nabla q_{it,\tau}(\boldsymbol{\theta}_{i\tau}|\boldsymbol{\theta}_{i\,0.5})$ *and* $\nabla q_{it,\tau}(\boldsymbol{\theta}_{i\,0.5})$ *denote the gradient of* $q_{it,\tau}(\boldsymbol{\theta}_{i\tau}|\boldsymbol{\theta}_{i\,0.5})$ *and* $q_{it,\tau}(\boldsymbol{\theta}_{i\,0.5})$ *with respect to* $\mathbf{\Theta}_{i\tau}$*, respectively. Let* $\widehat{\boldsymbol{\eta}}_t = \nabla q_{it,\tau}(\widehat{\boldsymbol{\theta}}_{i\tau}|\widehat{\boldsymbol{\theta}}_{i\,0.5})\psi_\tau(\widehat{\epsilon}_{it,\tau}) + \sum_{i=1}^{n}\nabla q_{it,0.5}(\widehat{\boldsymbol{\theta}}_{i\,0.5})\psi_{0.5}(\widehat{\epsilon}_{it,0.5})$ *where*

$$\widehat{\epsilon}_{it,\tau} = \begin{cases} y_{it} - q_{it,\tau}(\widehat{\boldsymbol{\theta}}_{i\tau}|\widehat{\boldsymbol{\theta}}_{i\,0.5}), & \textit{for}\;\; \tau \neq 0.5, \\ y_{it} - q_{it,0.5}(\widehat{\boldsymbol{\theta}}_{i\,0.5}), & \textit{for}\;\; \tau = 0.5. \end{cases}$$

[9] For $\tau \neq 0.5$, $\nabla q_{it}(\tau)$ is a $[n(1+n+p) + (1+n+2p)]$ by 1 vector. For $\tau = 0.5$, $\nabla q_{it}(\tau)$ is a $n(1+n+p)$ by 1 vector due to $d_{k,i\,0.5} = 0$.

Following WKM, $\mathbf{Q}$ *and* $\mathbf{V}$ *are consistently estimated as*

$$\widehat{\mathbf{Q}} = \frac{1}{T}\sum_{t=1}^{T}\Big[\frac{1}{2\hat{c}_{i\tau}}1_{[|\widehat{\epsilon}_{it,\tau}|\le \hat{c}_{i\tau}]}\nabla q_{it,\tau}(\widehat{\boldsymbol{\theta}}_{i\tau}|\widehat{\boldsymbol{\theta}}_{i\,0.5})\nabla q_{it,\tau}(\widehat{\boldsymbol{\theta}}_{i\tau}|\widehat{\boldsymbol{\theta}}_{i\,0.5})^{\top}+ \sum_{i=1}^{n}\frac{1}{2\hat{c}_{i0.5}}1_{[|\widehat{\epsilon}_{it,0.5}|\le \hat{c}_{i\,0.5}]}\nabla q_{it,0.5}(\widehat{\boldsymbol{\theta}}_{i\,0.5})\nabla q_{it,0.5}(\widehat{\boldsymbol{\theta}}_{i\,0.5})^{\top}\Big],$$

$$\widehat{\mathbf{V}} = \frac{1}{T}\sum_{t=1}^{T}\widehat{\boldsymbol{\eta}}_t\widehat{\boldsymbol{\eta}}_t^{\top},$$

where $\hat{c}_{i\tau}$ *is the same bandwidth defined in Machado and Santos Silva (2013) and WKM.*

5 Quantile Impulse Response Analysis of US Economy

In this section, We perform QIRF applications with US macroeconomic and financial data. Monetary policy has been one of the most heavily studied topics in macroeconomics, and the IRF is the main tool for evaluating its policy implications in VAR studies. Recently, some financial conditions indices have received much attention for explaining economic fluctuations.[10] The impact of financial shocks on the whole economy is now considered dominant after the 2007–2009 financial crisis. We estimate a QR model (2) with four variables including the federal funds rate and a financial conditions index. Then we investigate the effects of monetary policy and financial shocks on the quantile responses using QIRFs. In particular, we provide the dynamic responses of Growth-at-Risk (5% quantile of CFNAI) using QIRF.

5.1 Data

The variables under study are the CFNAI, the inflation rate (CPI), the federal funds rate (FFR), and the National Financial Conditions Index (NFCI). The CFNAI is a monthly index for US economic activity, released by the Federal Reserve Bank of Chicago. The index is a weighted average of 85 indicators of national economic activity and captures movements of the GDP growth well. For our sample period (1971Q1–2018Q4), the correlation between GDP growth rate and CFNAI is 0.73. The NFCI is a weekly index describing US financial conditions in the money market, debt and equity markets, and traditional and shadow banking systems. The index, also released by the Federal Reserve Bank of Chicago, is a weighted average of 105 indicators of national financial activity.[11]

For the sample period from January 1971 to December 2018, we use monthly data of the four variables. We measure inflation rate as the log difference of CPI, multiplied by 100. For the federal funds rate between 2009 and 2015 during which it reached the zero lower bound, we use the shadow

[10] See, e.g., Brave and Butters (2011), Matheson (2012), and Koop and Korobilis (2014).

[11] More details about the two indices are available at *https://www.chicagofed.org/publications/cfnai/index* (CFNAI) and *https://www.chicagofed.org/publications/nfci/index* (NFCI).

federal funds rate estimated by Wu and Xia (2016).[12] For the NFCI, we use its monthly average. All data are from Federal Reserve Economic Data (FRED).

5.2 Estimated Conditional Quantiles

Prior to investigating QIRFs, we examine how the conditional distribution of the four variables evolved over the sample period. We estimate the QR model (2) of lag order 2 for lagged conditional quantile terms. Figure 1 illustrates the estimated conditional 5% and 95% quantiles from the model for 2001–2015.[13] While the two quantiles co-move in each variable, they do not fluctuate in the same way. Due to their heterogeneous movements, the distance between the lower and upper quantiles (the dispersion of distribution) changes over time. For instance, the conditional distribution was more dispersed during the global financial crisis compared to other periods, in all variables.

However, the degree of heterogeneity in the quantile movements varies depending on given variables. Among the variables, dissimilar dynamics of the downside and upside risks are particularly pronounced in the CFNAI. Between February 2007 and 2009, for example, the conditional 5% quantile decreased by 4.8, but the 95% quantile decreased only by 2.9. In other variables, the movements of the two quantiles are not heterogeneous as much as in the CFNAI.

The summary statics for the estimated conditional 5% and 95% quantiles in Table 1 also support that evolution of the downside and upside risks are the most disparate in the CFNAI. The correlation coefficient between the two quantiles is the smallest in the CFNAI. In addition, its left tail shows a much larger time variation than its right tail. Measured by standard deviation, the variation of the 5% quantile is about 1.5 times as large as that of the 95% quantile. These findings suggest substantial heterogeneity in the QIRF of the CFNAI.

On the contrary, we expect the QIRF of the financial variables (FFR and NFCI) would be less heterogeneous than the CFNAI. As seen in Figure 1(c) and (d), the correlation between their conditional 5% and 95% quantiles is very strong: their correlation coefficients are close to one. The time variation of the left and right tails are not substantially different as in the CFNAI. The standard deviation of the 95% quantile is about 1.2 times as large as that of the 5% quantile in the variables.

In the CPI, the correlation coefficient between its 5% and 95% quantiles is modest: 0.72. The time variations of the downside and upside risks are not substantially different. Accordingly, a certain degree of heterogeneity in the QIRF of CPI is expected, but not as much as that of the CFNAI.

5.3 Quantile Impulse Response Analysis

We now construct QIRFs based on the QR model estimates. Since our QR model is a reduced form, we use a mean-based VAR model with Cholesky restrictions to identify a structural shock.

[12] The data is available at *https://sites.google.com/view/jingcynthiawu/shadow-rates*.

[13] The estimated conditional 5% and 95% quantiles for the whole sample period is in Appendix B.1

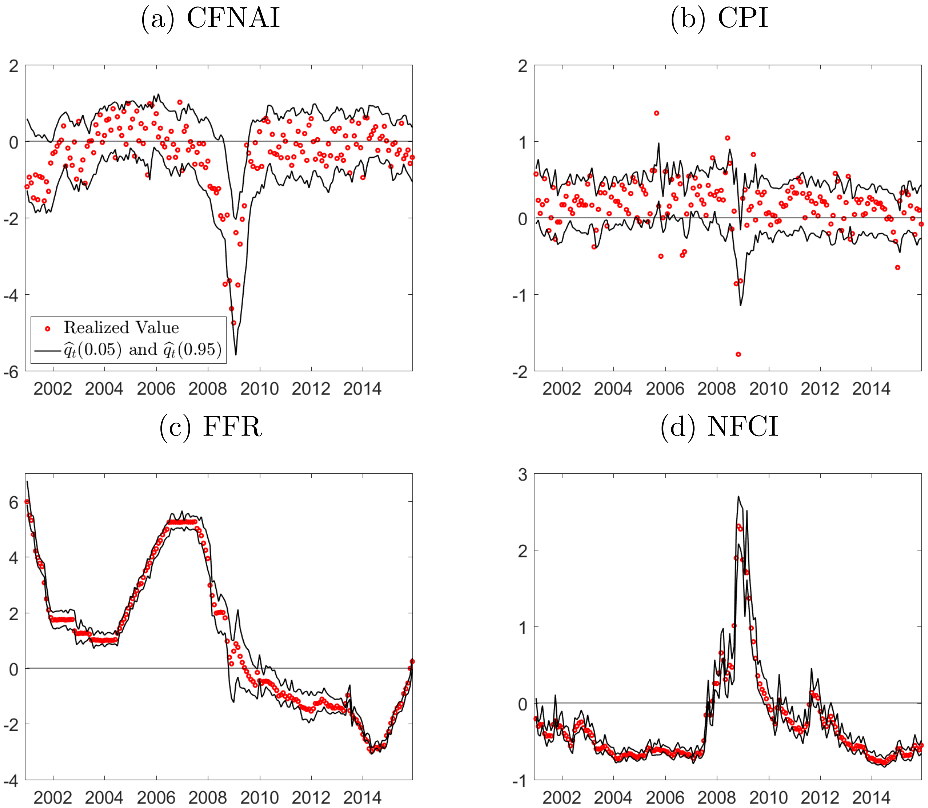


Figure 1: The Estimated Conditional 5% and 95% Quantiles for 2001–2015

Table 1: Summary Statistics for the Estimated Conditional 5% and 95% Quantiles

	CFNAI	CPI	FFR	NFCI
Correlation coefficient between $\widehat{q}_t(0.05)$ and $\widehat{q}_t(0.95)$	0.49	0.72	0.99	0.999
Standard deviation of $\widehat{q}_t(0.05)$	1.01	0.20	3.96	0.92
Standard deviation of $\widehat{q}_t(0.95)$	0.66	0.24	4.62	1.11

Under the recursive identification, a variable is affected by the contemporaneous shocks to other variables if the variable is ordered after them, but not affected if ordered before them. Thus, the slowly moving variables are ordered before the faster responding variables. The ordering of our variables (CFNAI, CPI, FFR, and NFCI, standard in the literature) implies that the NFCI instantly responds to all structural shocks. But, the CFNAI is not contemporaneously affected by shocks other than a shock to itself. Following the Bayesian information criterion (BIC), we estimate a VAR(3) model. The VAR model is stable since the largest eigenvalue of the companion matrix is strictly less than one.

5.3.1 QIRF to Monetary Policy Shock

First, we present how the conditional quantile of the variables responds to a monetary policy shock. Figure 2 is the QIRF to an expansionary monetary policy shock (-25bp) at five different quantiles

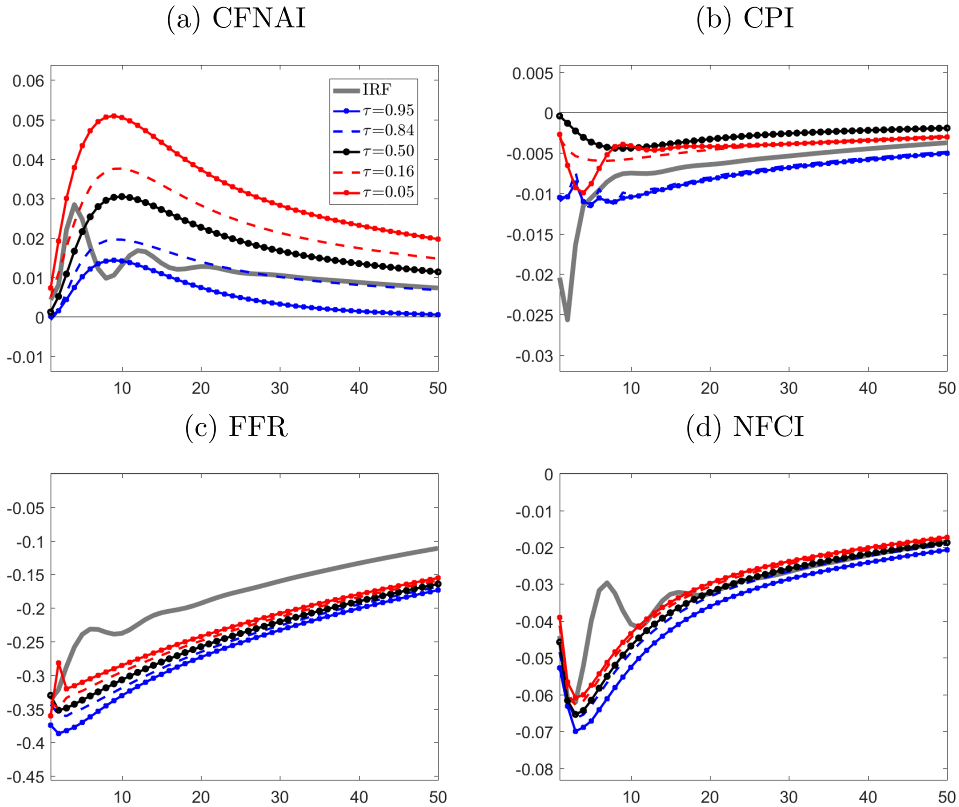


Figure 2: QIRF to a Monetary Policy Shock (-25bp)

(5%, 16%, 50%, 84%, and 95%) as well as the IRF.[14]

Similarly to IRF, the QIRF of CFNAI is positive at all of the five quantiles. But, the highly heterogeneous QIRF clearly describes how the distribution of economic activity shifts to the right. An expansionary monetary policy shock significantly reduces downside risk to growth, while upside risks are affected much less than that. Since quantiles closer to the left tail are more responsive, the dispersion of distribution is decreased by the shock. When the monotonic response along all the quantiles is preserved, the distribution after the shock first-order stochastically dominates the distribution before the shock.[15] Using local projections, Loria, Matthes, and Zhang (2019) also find that a monetary policy shock affects downside risk to growth more than its upside risk. However, our QIRF does not show the reverse effect of the shock at longer horizons as in their conclusion.

The conditional 5% quantile of economic activity has a much more persistent response to a monetary policy shock than the mean IRF of the economic activity. It takes 26 months for the 5% QIRF of CFNAI to dissipate by half from its peak, while it takes only 11 months for the IRF. These dynamics shows that the effects of a monetary policy shock on Growth-at-Risk last over longer horizons.

The quantile response of CPI, like its IRF, suggests that an increase in the FFR lowers prices–the

[14] In Appendix B.2, the QIRFs are provided with their asymptotic confidence intervals.

[15] See, e.g., Fomby and Seo (1989) for further discussion of stochastic dominance.

so-called price puzzle–in which an increase (decrease) in the price level in response to a contractionary (expansionary) monetary policy shock.[16] Though the magnitude of QIRF is small across quantiles (between -0.01% and 0%), the price puzzle is less pronounced in our QIRF compared to the IRF until horizon 4.

An expansionary monetary policy shock leads to looser financial conditions (negative response of NFCI), while having persistent effects on the FFR. In both variables, however, the QIRF at the considered five quantiles is more responsive than the IRF overall. This result is because our QIRF accounts for heteroskedastic behavior of financial variables, as explained in Section 3.2. An expansionary monetary policy shock reduces volatility of the FFR and the NFCI, as the upper quantiles decline more than the lower quantiles.

The financial variables have more homogeneous QIRFs than economic activity. We interpret the QIRFs as relatively stable responses of the Fed and financial markets: their response to a one-off monetary shock does not significantly increase tail risks to the FFR and financial conditions.

5.3.2 QIRF to Financial Shock

Figure 3 plots the QIRF and IRF to a financial shock. Similarly to the case of a monetary policy shock, the response of the CFNAI is highly heterogeneous across quantiles. An adverse financial shock shifts the conditional distribution of economic activity to the left, but the left tail quantiles decrease substantially more than the right tails.

This empirical result is in line with Adrian, Boyarchenko, and Giannone (2019): economic growth is vulnerable to deteriorating financial conditions. They argue that downside GDP vulnerability (away from the steady state) can be explained by amplification mechanisms in the financial sector, such as the feedback loops mechanism by Brunnermeier and Sannikov (2014). While they examine the impact of a financial shock on the next period's conditional quantile of economic growth, our QIRF further describes the evolution of the quantile response over time.

The effects of a financial shock are more acute but transient on the conditional 5% quantile of economic activity than on its mean. It takes 8 months for the 5% QIRF of CFNAI to decay by half from its trough, but 11 months for the IRF. Compared to a monetary policy shock, a financial shock creates severe, but less persistent, downside risk to growth.

In the FFR, the response of the 95% quantile is much different from that of the other quantiles. In response to a financial shock, the 95% quantile of the FFR increases by 7 basis points, on average, over 6-month horizons. But the response of the FFR at the other quantiles suggests that the Fed cut the rate against the shock. The IRF also shows positive initial response for 4 months after the shock. The upside risk to the FFR seems driven by observations in the 1970s. For a sample period from January 1981 to December 2018, the initial response of the FFR to a financial shock is not significantly positive for both the IRF and the QIRF at the 95% quantile. The different results from the two sample periods may stem from the change in monetary policy practice or the change

[16] See, e.g., Sims (1992), Christiano, Eichenbaum, and Evans (1999), and Hanson (2004) for further discussion of the price puzzle.

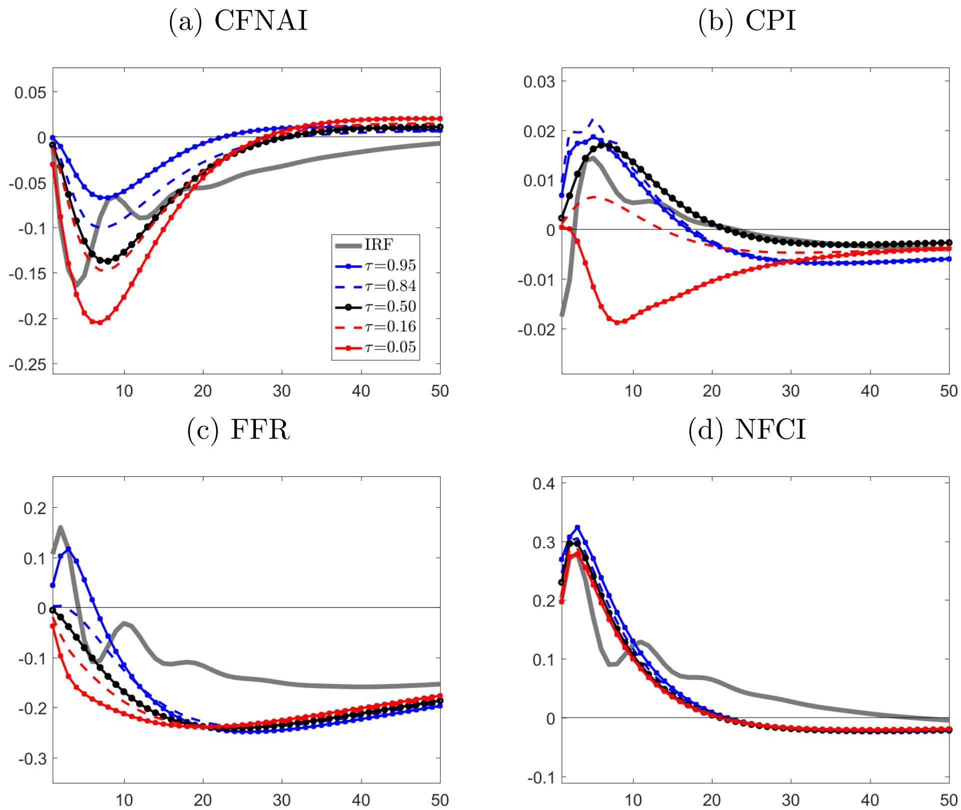


Figure 3: QIRF to a Financial Shock (one standard deviation shock)

in how the other variables respond to shocks.[17]

A financial shock tightens financial conditions for over a year. For all of the five quantiles considered, the positive response of NFCI is statistically significant until 16 months after the shock. The QIRF of NFCI shows different dynamics from the IRF. Compared to the IRF, the QIRF has greater initial responses for 9 months after the shock, but decays faster than the IRF.

CPI shows a certain degree of heterogeneity in its quantile response. But the magnitude of the QIRF is not very significant (between -0.2% and +0.2%) as in the case of a monetary policy shock.

5.4 Growth-at-Risk Dynamics during the Global Financial Crisis

In this section, we examine how the conditional quantile of economic activity is affected by a series of shocks during a particular historical episode: the 2007–2009 global financial crisis. We pay close attention to the conditional 5% quantile of the CFNAI, considered as *Growth-at-Risk* in this paper. The dynamic response of the quantile to a one-off shock is described by the QIRF. We now investigate the cumulative effects of a set of shocks on the quantile with an application of QIRFs. Since the CFNAI has a high degree of heterogeneity in its QIRF to monetary policy and financial shocks, we focus on its quantile response.

[17]See, e.g., Primiceri (2005) and Sims and Zha (2006) for a discussion of the change in monetary policy rules. We leave more rigorous investigation of these causal explanations to future research.

We provide an answer to the following question: what are the impacts of financial and monetary policy shocks concerning the global financial crisis on the downside and upside risks to growth? As seen in the previous section, the lower quantiles of economic activity are much more responsive than the median or upper quantiles. Thus, quantitative assessment of the downside risks during the recession period is of great importance. In the following exercises, we first study the impacts of financial shocks during the financial crisis, August 2007–June 2009. We then examine the effects of ensuing unconventional monetary policy measures, which are identified by the monetary policy shocks from July 2009 to December 2015.

The assessment is carried out in a similar way to historical decomposition in the VAR analysis.[18] For a given period, a series of structural shocks are identified based on the mean-based VAR model. As the dynamic effect of each shock is derived from the QIRF, we aggregate the impacts of those shocks accounting for dynamics. Suppose, for example, we want to quantify the effects of structural shocks from time 1 to T, $\{\boldsymbol{\epsilon}_t\}_{t=1}^{T}$, on the quantile response. Let $QIRF_{i\tau}^{(s)}|\boldsymbol{\epsilon}_t$ denote the τ-quantile response of the i-th variable to $\boldsymbol{\epsilon}_t$ at horizon s. Then, the cumulative impacts of $\{\boldsymbol{\epsilon}_t\}_{t=1}^{T}$ on the τ-quantile of the variable at time s is calculated as

$$\begin{cases} \sum_{t=1}^{s-1} QIRF_{i\tau}^{(s-t)}|\boldsymbol{\epsilon}_t, & \text{for} \quad s \leq T, \\ \sum_{t=1}^{T} QIRF_{i\tau}^{(s-t)}|\boldsymbol{\epsilon}_t, & \text{for} \quad s > T. \end{cases}$$

For estimation, we replace $QIRF_{i\tau}^{(s-t)}$ and $\boldsymbol{\epsilon}_t$ with their estimators, $\widehat{QIRF}_{i\tau}^{(S-t)}$ and $\widehat{\boldsymbol{\epsilon}}_t$.

The cumulative impacts of financial shocks from August 2007 to June 2009 on the conditional quantile of the CFNAI are illustrated in Figure 4(a).[19] Financial shocks during the period substantially increased the downside risk, while the upside risk was mildly impacted; over 2008–2009, the financial shocks decreased the conditional 5% quantile of the CFNAI by 1.7 on average. On the other hand, the impacts of those shocks on the 95% quantile are much less than on the 5% quantile. During the same period, the conditional 95% quantile is lowered by 0.5 on average.

We then answer the following counterfactual question: what would have happened if the financial shocks were shut down? Figure 4(b) and (c) describe the counterfactual paths of the conditional 5% and 95% quantiles based on the cumulative impacts. The counterfactual 5% quantile is much higher than the 5% quantile. In the absence of the financial shocks, the downside risks over 2008–2009 would have been minor: the average of the counterfactual 5% quantile path over 2008–2009, -1.2, is a little less than the average of the 5% quantile path over 1971-2007: -1.0. On the contrary, the difference between the 95% quantile and its counterfactual is considerably less.

We perform the same exercise to quantify the effects of unconventional monetary policy measures implemented by the Fed in response to the financial crisis. Following Wu and Xia (2016), structural

[18]See Kilian and Lütkepohl (2017) for details about historical decomposition in the VAR model.

[19] In August 2007, BNP Paribas halted redemptions on three investment funds because it could not value their holdings, which marked the start of the financial crisis. The National Bureau of Economic Research (NBER) identified June 2009 as the end of the recession associated with the financial crisis.

(a) Impacts across Quantiles (b) Counterfactual $\widehat{q}_t(0.05)$ (c) Counterfactual $\widehat{q}_t(0.95)$

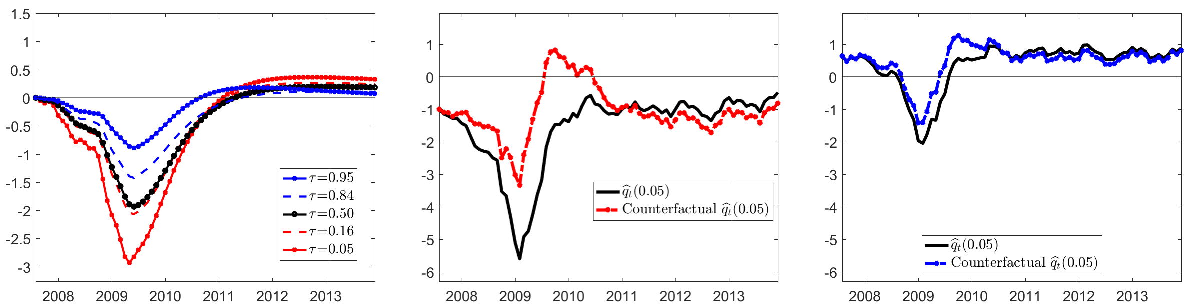


Figure 4: The Impacts of Financial Shocks (Oct.2007–Jun.2009) on CFNAI

Note: The counterfactual path describes the quantile path of CFNAI if the financial shocks were shut down.

shocks to the FFR from July 2009 to December 2015 are used for assessment of the measures.[20] Figure 5(a) illustrates the cumulative impacts of those monetary policy shocks on the conditional quantile of the CFNAI. The figure highlights the effectiveness of unconventional monetary policy tools for reducing the downside risks to growth. Over 2010–2015, the tools increased the 5% quantile of CFNAI by 0.5, on average. Meanwhile, the 95% quantile is increased by just 0.1.

The counterfactual paths in Figure 5(b) and (c) emphasize the asymmetric effects on economic growth. Without the unconventional monetary measures, downside risks would have been higher: the averages of the 5% quantile and its counterfactual over 2010–2015 are -1.4 and -0.9, respectively. The effects of monetary policy tools were fairly consistent over the period considered. However, the 95% quantile and its counterfactual are almost indistinguishable.

5.5 Quantile Responses Based on a Hypothetical Distress Scenario

The QIRF defined in Section 3.2 measures the impact of a one-time shock assuming realizations of endogenous variables at the conditional median after the shock. However, we can modify the assumed realizations to study the quantile response under a specific scenario.

In this section, we conduct a hypothetical analysis to examine the quantile response under a distress scenario in which a series of unfavorable tail events follow an initial shock. This analysis provides a tool for testing the resilience of the economy. In the exercises below, we consider two scenarios: one for a contractionary monetary policy shock and the other for a financial shock. The exercises share that persistent realizations of unlikely events are assumed under a hypothetical scenario with the stress testing of Chavleishvili and Manganelli (2019) and the QIRFs of Montes-Rojas (2019).

In the first exercise, the effects of a contractionary monetary policy shock (+25bp) are examined. For the first 3 months after the shock, realizations of the conditional 95% are assumed for the

[20] Wu and Xia (2016) study the effects of unconventional policy measures using monetary policy shocks from July 2009 to December 2013. We extend the period to December 2015, the end of the zero lower bound period.

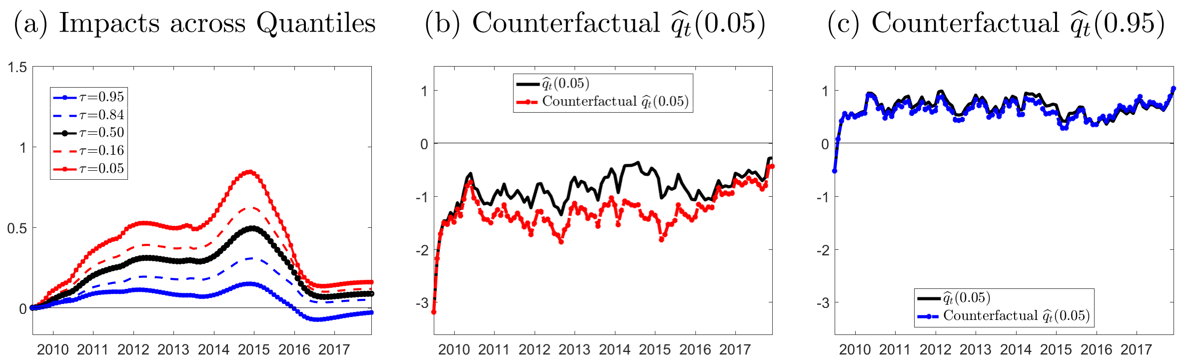


Figure 5: The Effects of Unconventional Monetary Policy (Jul.2009–Dec.2015) on CFNAI

Note: The counterfactual path describes the quantile path of CFNAI if the monetary policy shocks were shut down.

FFR and the NFCI. Afterwards, realizations of the conditional median are assumed for the two variables. For the CFNAI and the CPI, meanwhile, we assume every realization after the shock is the conditional median. The distress scenario describes a rapid increase in the FFR, which accompanies adverse financial conditions. The quantile response under this distress scenario can be expressed in a recursive manner as well, and its derivation is in Appendix A.3.

Figure 6 describes the quantile response under the first distress scenario. Since the FFR and the NFCI have explosive dynamics at their 95% conditional quantile, the distress scenario drastically shifts the distribution of the two financial variables to the right. The IRF of the FFR and the NFCI against a one-time monetary policy shock (+25bp) reaches its highest of +34bp and +0.06, respectively. Under the distress scenario, however, the conditional median of the FFR and the NFCI increases by up to 54bp and 0.12, respectively. Since those realizations of tail events act as a series of shocks to the FFR and the NFCI, the distribution of financial variables changes radically.

These movements of financial variables intensify the degree of heterogeneity in the quantile response of economic activity. Its conditional 5% quantile decreases as much as 0.10, whereas its 95% quantile decreases by 0.03. Consequently, the conditional distribution of the CFNAI becomes highly volatile compared to a one-time shock environment. For the CPI, on the contrary, the magnitude of the quantile response is small.

In the second exercise, we examine a distress scenario initiated by a financial shock. In this scenario, one standard deviation financial shock is followed by realizations of the NFCI at its conditional 95% quantile for the first 3 months. Since then, realizations of its conditional median are assumed. For variables other than the NFCI, realizations of the conditional median are assumed after the initial shock.

The quantile response under the second distress scenario is Figure 7. Due to its highly explosive dynamics at the 95% quantile, financial conditions deteriorate rapidly. Under the distress scenario, the median of the NFCI increases by up to 0.59, whereas a one-time shock increase its IRF by up to 0.28. By comparing Figure 3(d) and 7(d), it is observed that the consecutive tail events lead

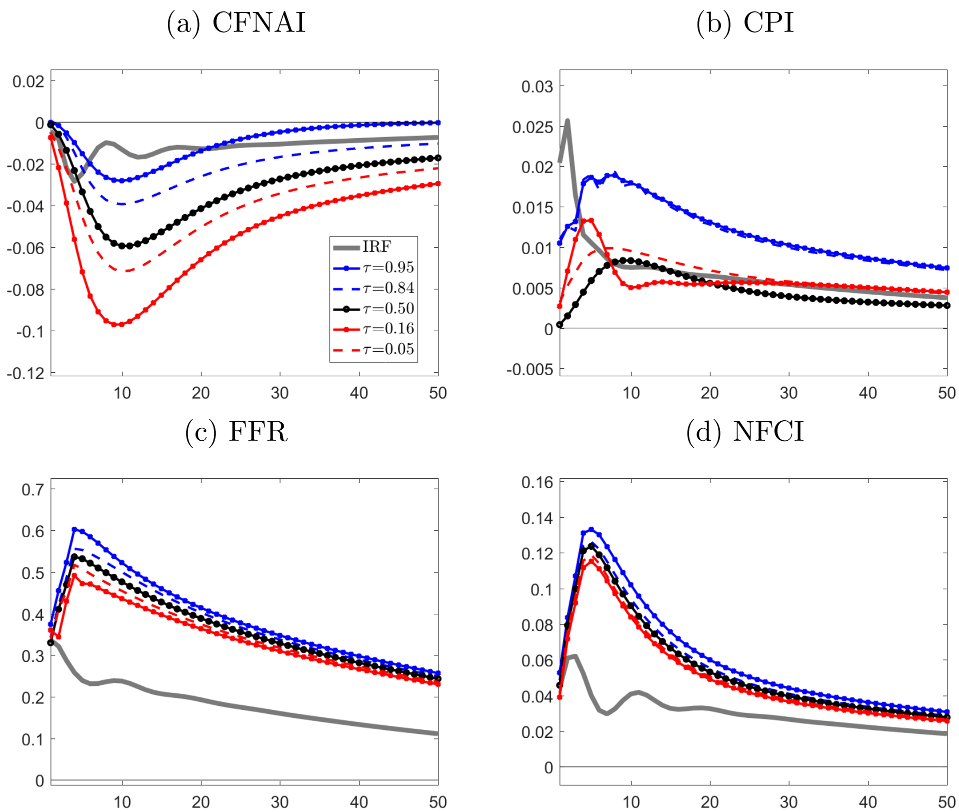


Figure 6: Quantile Responses Under the First Distress Scenario

Note: 1) After a shock of +25bp to the FFR, realizations of the conditional median are assumed for the CFNAI and the CPI. For the FFR and the NFCI, realizations of the conditional 95% follow for the first 3 months after the shock. Afterwards, realizations of the conditional median of are assumed for the two variables. 2) The IRF is based on the initial one-time shock.

to an acute financial tightening. This explosive behavior of financial conditions indicates that a financial crisis can develop in a short period of time.

As a result of the financial instability, the economic activity distribution substantially shifts to the left. In particular, the left tail decreases more than the median or upper quantiles. The 5% quantile of the CFNAI reaches its lowest of -0.44. In response to a series of the tail events, the considered five quantiles of FFR decrease by almost 50bp 18 months after the initial shock.

6 Conclusion

This article studies Quantile Impulse Response Function (QIRF) theory and its applications in macroeconomics and finance. Our QR model provides a multi-equation system with autoregressive specifications to account for the important dynamic evolution of distribution. The resulting QIRF can complement the conventional IRF by providing a much more complete shock-response mechanism.

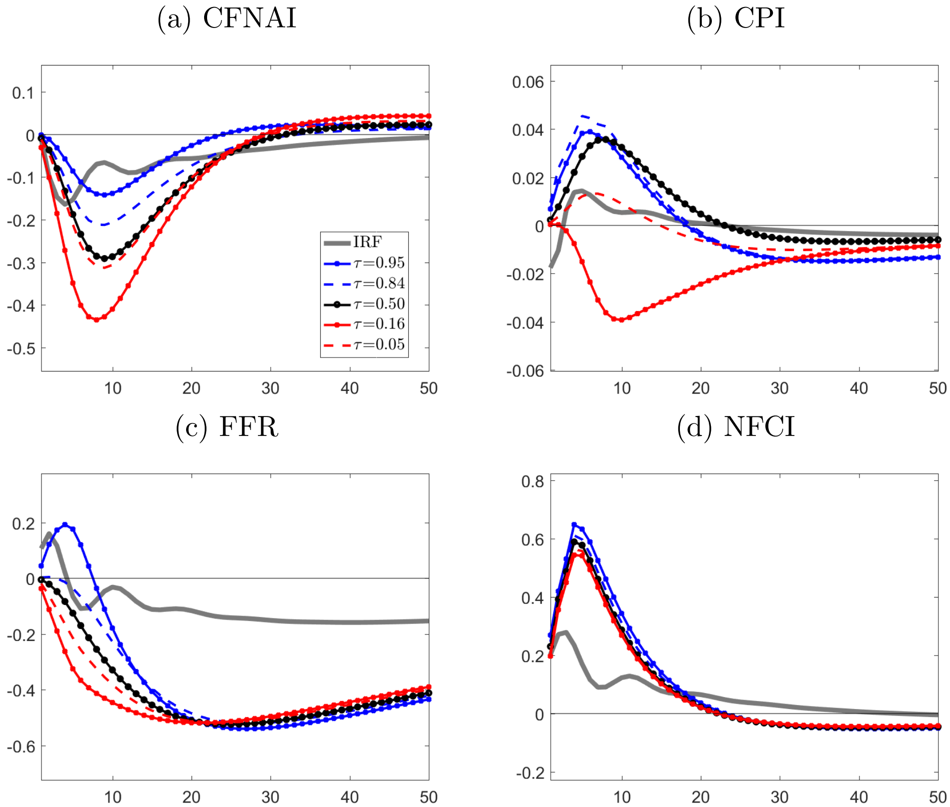


Figure 7: Quantile Responses Under the Second Distress Scenario

Note: 1) After a shock of one standard deviation to the NFCI, realizations of the conditional median are assumed for variables other than the NFCI. For the NFCI, realizations of the conditional 95% follow for the first 3 months after the shock. Afterwards, realizations of the conditional median are assumed for the variable. 2) The IRF is based on the initial one-time shock.

The comprehensive QIRF analysis of the US economy provides evidence of a strong heterogeneity in the responses of economic activity across its distribution. Against monetary policy and financial shocks, the downside risks to growth are more responsive than the median or upside risks. We also quantitatively assess how much the downside risks are affected by financial shocks and unconventional monetary tools during the 2007–2009 global financial crisis. Considering the tremendous implications of extreme events, such as market booms and crashes, our QR model and QIRF can provide essential tools in dynamic risk management and policy analysis.

The current article also suggests some future research directions of econometric methodology. One interesting question is how to relate the recent development in structural VAR identification to our QIRF model. In this paper, the QIRF and IRF are from two separate models; the QR model and the mean-based VAR model. One future research direction would be to develop an econometric tool that can systematically harmonize the QIRF and IRF constructions.

A Technical Appendix

A.1 The derivation of $QIRF_\tau^{(s)}$ in section 3.2

For horizon 1,

$$\begin{aligned}
QIRF_\tau^{(1)} &= Q_{\mathbf{y}_{t+1}}\big(\tau \mid \tilde{\mathbf{y}}_t; \mathcal{F}_{t-1}\big) - Q_{\mathbf{y}_{t+1}}\big(\tau \mid \mathbf{y}_t; \mathcal{F}_{t-1}\big) \\
&= \Big[\mathbf{c}(\tau) + \mathbf{A}(\tau)\tilde{\mathbf{y}}_t + \sum_{k=1}^{p}[\mathbf{B}_k(\tau)Q_{\mathbf{y}_{t+1-k}}(0.5 \mid \mathcal{F}_{t-1}) + \mathbf{D}_k(\tau)Q_{\mathbf{y}_{t+1-k}}(\tau \mid \mathcal{F}_{t-1})]\Big] \\
&\quad - \Big[\mathbf{c}(\tau) + \mathbf{A}(\tau)\mathbf{y}_t + \sum_{k=1}^{p}[\mathbf{B}_k(\tau)Q_{\mathbf{y}_{t+1-k}}(0.5 \mid \mathcal{F}_{t-1}) + \mathbf{D}_k(\tau)Q_{\mathbf{y}_{t+1-k}}(\tau \mid \mathcal{F}_{t-1})]\Big] \\
&= \mathbf{A}(\tau)[\tilde{\mathbf{y}}_t - \mathbf{y}_t] \\
&= \mathbf{A}(\tau)\boldsymbol{\epsilon}_t.
\end{aligned}$$

For horizon $s \geq 2$,

$$\begin{aligned}
QIRF_\tau^{(s)} &= Q_{\mathbf{y}_{t+s}}\big(\tau \mid \mathcal{M}_{t+s-1}(\tilde{\mathbf{y}}_t), \ldots, \mathcal{M}_{t+1}(\tilde{\mathbf{y}}_t), \tilde{\mathbf{y}}_t; \mathcal{F}_{t-1}\big) - Q_{\mathbf{y}_{t+s}}\big(\tau \mid \mathcal{M}_{t+s-1}(\mathbf{y}_t), \ldots, \mathcal{M}_{t+1}(\mathbf{y}_t), \mathbf{y}_t; \mathcal{F}_{t-1}\big) \\
&= \Big[\mathbf{c}(\tau) + \mathbf{A}(\tau)\mathcal{M}_{t+s-1}(\tilde{\mathbf{y}}_t) + \sum_{k=1}^{p}\big[\mathbf{B}_k(\tau)Q_{\mathbf{y}_{t+s-k}}\big(0.5 \mid \mathcal{M}_{t+s-k-1}(\tilde{\mathbf{y}}_t), \ldots, \mathcal{M}_{t+1}(\tilde{\mathbf{y}}_t), \tilde{\mathbf{y}}_t; \mathcal{F}_{t-1}\big) \\
&\qquad + \mathbf{D}_k(\tau)Q_{\mathbf{y}_{t+s-k}}\big(\tau \mid \mathcal{M}_{t+s-k-1}(\tilde{\mathbf{y}}_t), \ldots, \mathcal{M}_{t+1}(\tilde{\mathbf{y}}_t), \tilde{\mathbf{y}}_t; \mathcal{F}_{t-1}\big)\big]\Big] \\
&\quad - \Big[\mathbf{c}(\tau) + \mathbf{A}(\tau)\mathcal{M}_{t+s-1}(\mathbf{y}_t) + \sum_{k=1}^{p}\big[\mathbf{B}_k(\tau)Q_{\mathbf{y}_{t+s-k}}\big(0.5 \mid \mathcal{M}_{t+s-k-1}(\mathbf{y}_t), \ldots, \mathcal{M}_{t+1}(\mathbf{y}_t), \mathbf{y}_t; \mathcal{F}_{t-1}\big) \\
&\qquad + \mathbf{D}_k(\tau)Q_{\mathbf{y}_{t+s-k}}\big(\tau \mid \mathcal{M}_{t+s-k-1}(\mathbf{y}_t), \ldots, \mathcal{M}_{t+1}(\mathbf{y}_t), \mathbf{y}_t; \mathcal{F}_{t-1}\big)\big]\Big] \\
&= \mathbf{A}(\tau)[\mathcal{M}_{t+s-1}(\tilde{\mathbf{y}}_t) - \mathcal{M}_{t+s-1}(\mathbf{y}_t)] + \sum_{k=1}^{p}[\mathbf{B}_k(\tau)QIRF_{0.5}^{(s-k)} + \mathbf{D}_k(\tau)QIRF_\tau^{(s-k)}] \\
&= \mathbf{A}(\tau)QIRF_{0.5}^{(s-1)} + \sum_{k=1}^{r}[\mathbf{B}_k(\tau)QIRF_{0.5}^{(s-k)} + \mathbf{D}_k(\tau)QIRF_\tau^{(s-k)}]
\end{aligned}$$

where $r = \min\{s-1, p\}$.

A.2 The Functional Form of $\mathbf{G}_{i\tau}^{(s)}$ in Section 4.2

For $s = 1$,

$$\frac{\partial QIRF_{i\tau}^{(1)}}{\partial \boldsymbol{\Theta_i}^\top} = \frac{\partial \mathbf{a}_{i\tau}^\top \boldsymbol{\epsilon}_t}{\partial \boldsymbol{\Theta_i}^\top} = \boldsymbol{\epsilon}_t^\top \frac{\partial \mathbf{a}_{i\tau}}{\partial \boldsymbol{\Theta_i}^\top}.$$

For $s \geq 2$,

$$\frac{\partial QIRF_{i\tau}^{(s)}}{\partial \boldsymbol{\Theta_i}^\top} = \frac{\partial(\mathbf{a}_{i\tau}^\top QIRF_{0.5}^{(s-1)})}{\partial \boldsymbol{\Theta_i}^\top} + \sum_{k=1}^{r} \Big[\frac{\partial(b_{k,i\tau} QIRF_{i\,0.5}^{(s-k)})}{\partial \boldsymbol{\Theta_i}^\top} + \frac{\partial(d_{k,i\tau} QIRF_{i\tau}^{(s-k)})}{\partial \boldsymbol{\Theta_i}^\top}\Big]$$

where

$$\begin{aligned}
\frac{\partial(\mathbf{a}_{i\tau}^\top QIRF_{0.5}^{(s-1)})}{\partial \boldsymbol{\Theta_i}^\top} &= (QIRF_{0.5}^{(s-1)})^\top \frac{\partial \mathbf{a}_{i\tau}}{\partial \boldsymbol{\Theta_i}^\top} + \mathbf{a}_{i\tau}^\top \frac{\partial QIRF_{0.5}^{(s-1)}}{\partial \boldsymbol{\Theta_i}^\top} \\
\frac{\partial(b_{k,i\tau} QIRF_{i\,0.5}^{(s-k)})}{\partial \boldsymbol{\Theta_i}^\top} &= QIRF_{i\,0.5}^{(s-k)} \frac{\partial b_{k,i\tau}}{\partial \boldsymbol{\Theta_i}^\top} + b_{k,i\tau} \frac{\partial QIRF_{i\,0.5}^{(s-k)}}{\partial \boldsymbol{\Theta_i}^\top} \\
\frac{\partial(d_{k,i\tau} QIRF_{i\,\tau}^{(s-k)})}{\partial \boldsymbol{\Theta_i}^\top} &= QIRF_{i\tau}^{(s-k)} \frac{\partial d_{k,i\tau}}{\partial \boldsymbol{\Theta_i}^\top} + d_{k,i\tau} \frac{\partial QIRF_{i\,\tau}^{(s-k)}}{\partial \boldsymbol{\Theta_i}^\top}
\end{aligned}$$

and $r = \min\{s-1, p\}$.

A.3 The Derivation of Quantile Response under a Distress Scenario in Section 5.5

Let us consider a distress scenario in which a set of variables, D, are subject to unfavorable tail events after a shock at time t. After shock $\boldsymbol{\epsilon}_t$, realizations of the conditional median are assumed for variable $i \notin D$. For variable $i \in D$, however, realizations of the conditional τ quantile follow for the first k period after the shock. Afterwards, realizations of the conditional median are assumed.

Define

$$\widetilde{\mathcal{M}}_{i,t+s}(\mathbf{y}_t) := \begin{cases} Q_{\mathbf{y}_{i,t+s}}\big(\tau \mid \widetilde{\mathcal{M}}_{t+s-1}(\mathbf{y}_t),\ \dots\ , \widetilde{\mathcal{M}}_{t+1}(\mathbf{y}_t), \mathbf{y}_t; \mathcal{F}_{t-1}\big), & \text{for } i \in D \text{ and } 1 \leq s \leq k, \\ Q_{\mathbf{y}_{i,t+s}}\big(0.5 \mid \widetilde{\mathcal{M}}_{t+s-1}(\mathbf{y}_t),\ \dots\ , \widetilde{\mathcal{M}}_{t+1}(\mathbf{y}_t), \mathbf{y}_t; \mathcal{F}_{t-1}\big), & \text{otherwise.} \end{cases}$$

As in the definition of QIRF, the quantile response under the distress scenario can be measured by comparing the conditional quantiles from the following two time paths:

$$\begin{array}{lllllllll}
\{\ \dots, & \mathbf{y}_{t-2} & \mathbf{y}_{t-1}, & \mathbf{y}_t, & \widetilde{\mathcal{M}}_{t+1}(\mathbf{y}_t) & \widetilde{\mathcal{M}}_{t+2}(\mathbf{y}_t) & \widetilde{\mathcal{M}}_{t+3}(\mathbf{y}_t) & \dots\ \}, \\
\{\ \dots, & \mathbf{y}_{t-2} & \mathbf{y}_{t-1}, & \tilde{\mathbf{y}}_t, & \widetilde{\mathcal{M}}_{t+1}(\tilde{\mathbf{y}}_t) & \widetilde{\mathcal{M}}_{t+2}(\tilde{\mathbf{y}}_t) & \widetilde{\mathcal{M}}_{t+3}(\tilde{\mathbf{y}}_t) & \dots\ \},
\end{array} \tag{8}$$

where $\tilde{\mathbf{y}}_t = \mathbf{y}_t + \boldsymbol{\epsilon}_t$.

Let $\widetilde{QIRF}_{\tau}^{(s)}$ denote the τ-quantile response of endogenous variables to $\boldsymbol{\epsilon}_t$ under the distress scenario, at horizon s. Then, $\widetilde{QIRF}_{\tau}^{(s)}$ can be expressed in a recursive manner as the following:

For horizon 1,

$$\begin{aligned}
\widetilde{QIRF}_{\tau}^{(1)} &= Q_{\mathbf{y}_{t+1}}\big(\tau \mid \tilde{\mathbf{y}}_t; \mathcal{F}_{t-1}\big) - Q_{\mathbf{y}_{t+1}}\big(\tau \mid \mathbf{y}_t; \mathcal{F}_{t-1}\big) \\
&= \big[\mathbf{c}(\tau) + \mathbf{A}(\tau)\tilde{\mathbf{y}}_t + \sum_{k=1}^{p}[\mathbf{B}_k(\tau)Q_{\mathbf{y}_{t+1-k}}(0.5 \mid \mathcal{F}_{t-1}) + \mathbf{D}_k(\tau)Q_{\mathbf{y}_{t+1-k}}(\tau \mid \mathcal{F}_{t-1})]\big] \\
&\quad - \big[\mathbf{c}(\tau) + \mathbf{A}(\tau)\mathbf{y}_t + \sum_{k=1}^{p}[\mathbf{B}_k(\tau)Q_{\mathbf{y}_{t+1-k}}(0.5 \mid \mathcal{F}_{t-1}) + \mathbf{D}_k(\tau)Q_{\mathbf{y}_{t+1-k}}(\tau \mid \mathcal{F}_{t-1})]\big] \\
&= \mathbf{A}(\tau)[\tilde{\mathbf{y}}_t - \mathbf{y}_t] \\
&= \mathbf{A}(\tau)\boldsymbol{\epsilon}_t.
\end{aligned}$$

For horizon $2 \leq s \leq k+1$,

$$\begin{aligned}
\widetilde{QIRF}_{\tau}^{(s)} &= Q_{\mathbf{y}_{t+s}}\big(\tau \mid \widetilde{\mathcal{M}}_{t+s-1}(\tilde{\mathbf{y}}_t), \ldots, \widetilde{\mathcal{M}}_{t+1}(\tilde{\mathbf{y}}_t), \tilde{\mathbf{y}}_t; \mathcal{F}_{t-1}\big) - Q_{\mathbf{y}_{t+s}}\big(\tau \mid \widetilde{\mathcal{M}}_{t+s-1}(\mathbf{y}_t), \ldots, \widetilde{\mathcal{M}}_{t+1}(\mathbf{y}_t), \mathbf{y}_t; \mathcal{F}_{t-1}\big) \\
&= \Big[\mathbf{c}(\tau) + \mathbf{A}(\tau)\widetilde{\mathcal{M}}_{t+s-1}(\tilde{\mathbf{y}}_t) + \sum_{k=1}^{p}\big[\mathbf{B}_k(\tau)Q_{\mathbf{y}_{t+s-k}}\big(0.5 \mid \widetilde{\mathcal{M}}_{t+s-1}(\tilde{\mathbf{y}}_t), \ldots, \widetilde{\mathcal{M}}_{t+1}(\tilde{\mathbf{y}}_t), \tilde{\mathbf{y}}_t; \mathcal{F}_{t-1}\big) \\
&\qquad + \mathbf{D}_k(\tau)Q_{\mathbf{y}_{t+s-k}}\big(\tau \mid \widetilde{\mathcal{M}}_{t+s-1}(\tilde{\mathbf{y}}_t), \ldots, \widetilde{\mathcal{M}}_{t+1}(\tilde{\mathbf{y}}_t), \tilde{\mathbf{y}}_t; \mathcal{F}_{t-1}\big)\big]\Big] \\
&\quad - \Big[\mathbf{c}(\tau) + \mathbf{A}(\tau)\widetilde{\mathcal{M}}_{t+s-1}(\mathbf{y}_t) + \sum_{k=1}^{p}\big[\mathbf{B}_k(\tau)Q_{\mathbf{y}_{t+s-k}}\big(0.5 \mid \widetilde{\mathcal{M}}_{t+s-1}(\mathbf{y}_t), \ldots, \widetilde{\mathcal{M}}_{t+1}(\mathbf{y}_t), \mathbf{y}_t; \mathcal{F}_{t-1}\big) \\
&\qquad + \mathbf{D}_k(\tau)Q_{\mathbf{y}_{t+s-k}}\big(\tau \mid \widetilde{\mathcal{M}}_{t+s-1}(\mathbf{y}_t), \ldots, \widetilde{\mathcal{M}}_{t+1}(\mathbf{y}_t), \mathbf{y}_t; \mathcal{F}_{t-1}\big)\big]\Big] \\
&= \mathbf{A}(\tau)[\widetilde{\mathcal{M}}_{t+s-1}(\tilde{\mathbf{y}}_t) - \widetilde{\mathcal{M}}_{t+s-1}(\mathbf{y}_t)] + \sum_{k=1}^{p}[\mathbf{B}_k(\tau)\widetilde{QIRF}_{0.5}^{(s-k)} + \mathbf{D}_k(\tau)\widetilde{QIRF}_{\tau}^{(s-k)}] \\
&= \mathbf{A}(\tau)[\sum_{i=1}^{n} e_i(1_{[i\in D]}\widetilde{QIRF}_{i\tau}^{(s)} + 1_{[i\notin D]}\widetilde{QIRF}_{i\,0.5}^{(s)})] + \sum_{k=1}^{p}[\mathbf{B}_k(\tau)\widetilde{QIRF}_{0.5}^{(s-k)} + \mathbf{D}_k(\tau)\widetilde{QIRF}_{\tau}^{(s-k)}],
\end{aligned}$$

where $r = \min\{s-1, p\}$ and e_i is the $n \times 1$ unit vector with the i-th element equal to one and the rest zero.

Likewise, for horizon $s > k+1$,

$$\begin{aligned}
\widetilde{QIRF}_{\tau}^{(s)} &= Q_{\mathbf{y}_{t+s}}\big(\tau \mid \tilde{\mathbf{y}}_{t+s-1}, \ldots, \tilde{\mathbf{y}}_{t+1}, \tilde{\mathbf{y}}_t; \mathcal{F}_{t-1}\big) - Q_{\mathbf{y}_{t+s}}\big(\tau \mid \mathcal{M}_{t+s-1}(\mathbf{y}_t), \ldots, \mathcal{M}_{t+1}(\mathbf{y}_t), \mathbf{y}_t; \mathcal{F}_{t-1}\big) \\
&= \mathbf{A}(\tau)[Q_{\mathbf{y}_{t+s-1}}\big(0.5 \mid \tilde{\mathbf{y}}_{t+s-2}, \ldots, \tilde{\mathbf{y}}_{t+1}, \tilde{\mathbf{y}}_t; \mathcal{F}_{t-1}\big) - \mathcal{M}_{t+s-1}(\mathbf{y}_t)] \\
&\qquad + \sum_{k=1}^{p}[\mathbf{B}_k(\tau)\widetilde{QIRF}_{0.5}^{(s-k)} + \mathbf{D}_k(\tau)\widetilde{QIRF}_{\tau}^{(s-k)}] \\
&= \mathbf{A}(\tau)\widetilde{QIRF}_{0.5}^{(s-1)} + \sum_{k=1}^{r}[\mathbf{B}_k(\tau)\widetilde{QIRF}_{0.5}^{(s-k)} + \mathbf{D}_k(\tau)\widetilde{QIRF}_{\tau}^{(s-k)}]
\end{aligned}$$

where $r = \min\{s-1, p\}$.

B Figures

B.1 Estimated Conditional Quantiles

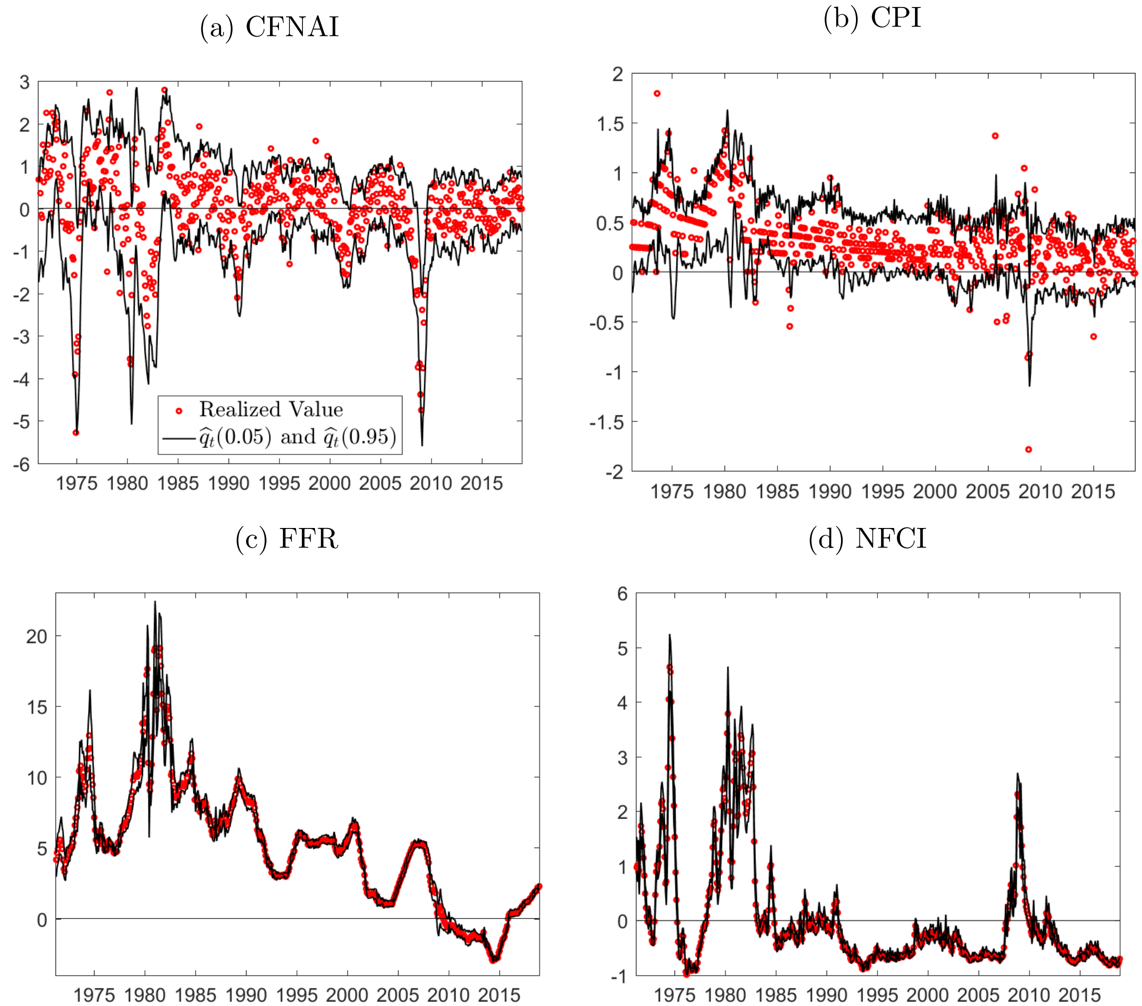


Figure 8: The Estimated Conditional 5% and 95% Quantiles for 1971–2018

B.2 QIRFs to Monetary Policy and Financial Shocks with Asymptotic Confidence Intervals

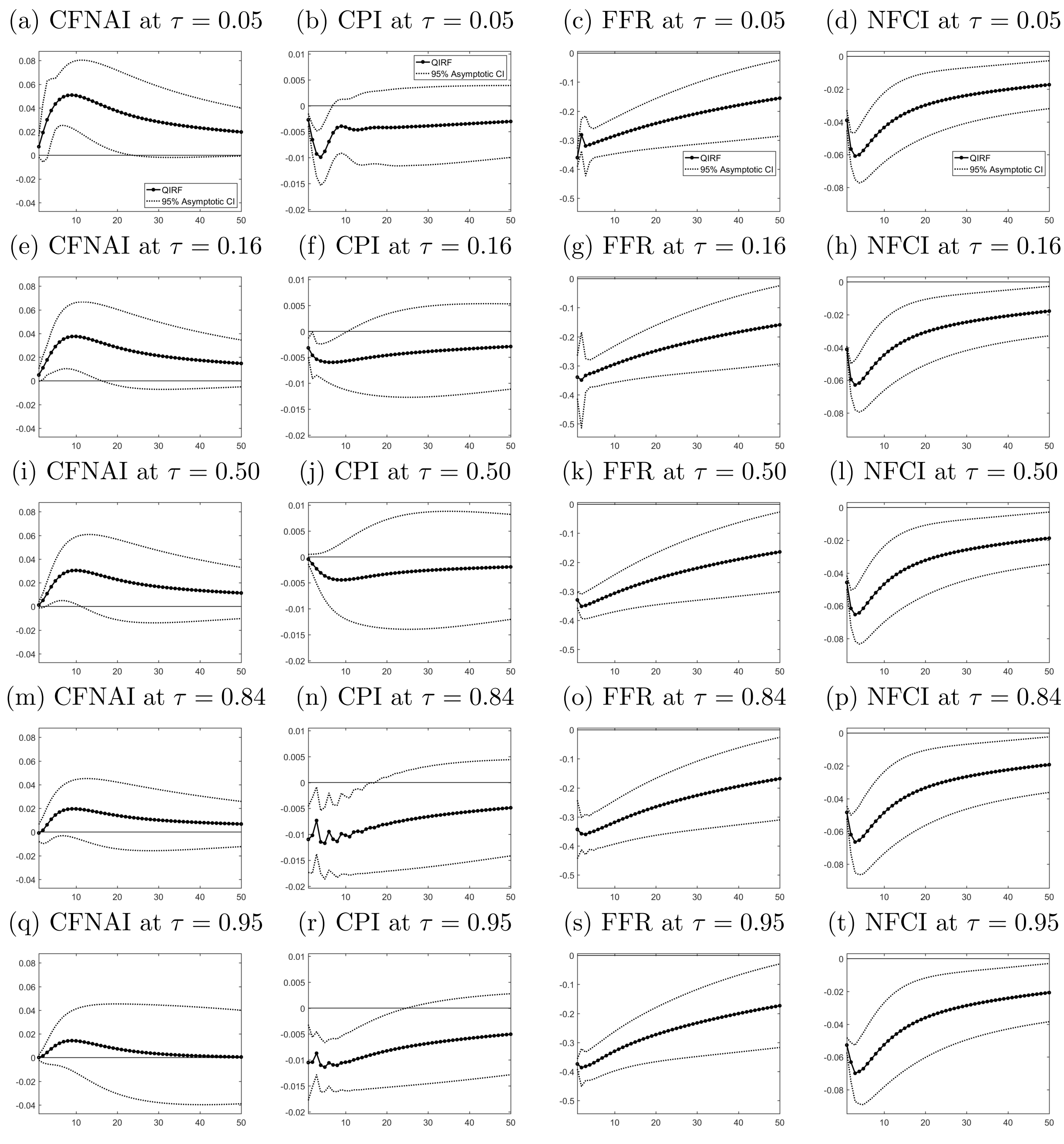
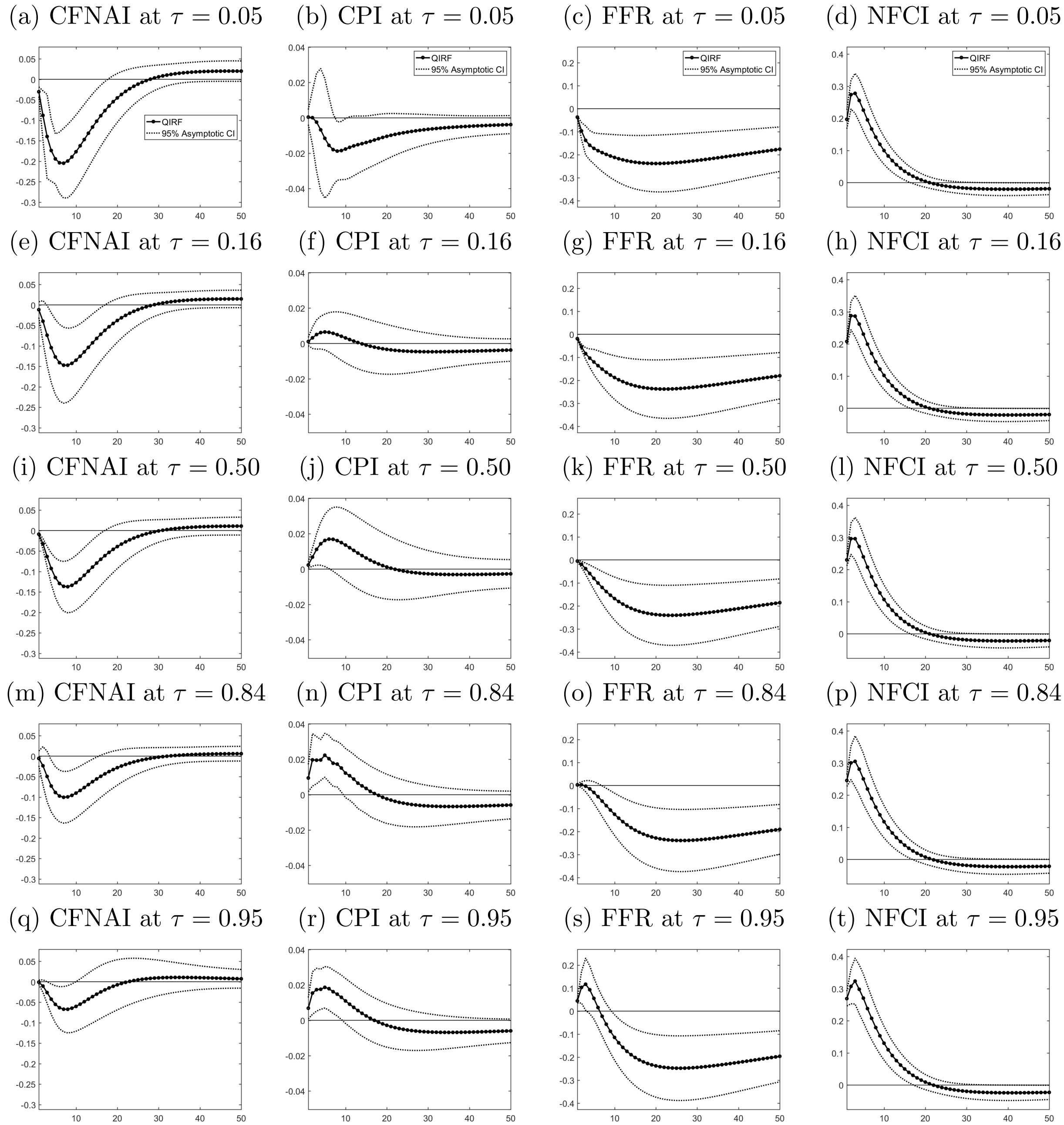


Figure 9: QIRF to a Monetary Policy Shock (-25bp)



(a) CFNAI at $\tau = 0.05$ (b) CPI at $\tau = 0.05$ (c) FFR at $\tau = 0.05$ (d) NFCI at $\tau = 0.05$

(e) CFNAI at $\tau = 0.16$ (f) CPI at $\tau = 0.16$ (g) FFR at $\tau = 0.16$ (h) NFCI at $\tau = 0.16$

(i) CFNAI at $\tau = 0.50$ (j) CPI at $\tau = 0.50$ (k) FFR at $\tau = 0.50$ (l) NFCI at $\tau = 0.50$

(m) CFNAI at $\tau = 0.84$ (n) CPI at $\tau = 0.84$ (o) FFR at $\tau = 0.84$ (p) NFCI at $\tau = 0.84$

(q) CFNAI at $\tau = 0.95$ (r) CPI at $\tau = 0.95$ (s) FFR at $\tau = 0.95$ (t) NFCI at $\tau = 0.95$

Figure 10: QIRF to a Financial Shock

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